Chiral symmetry and light-front zero modes: state of the art and a fresh look

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Abstract

A brief review of past light-front (LF) approaches to chiral-symmetry breaking and to LF zero modes is presented. A quantum-mechanical treatment of the dynamical zero modes and their corresponding vacuum potential leading to "measurable" effects in a two-dimensional non-abelian model will be briefly discussed. Then a novel second-quantized picture of the dynamical LF zero modes will be formulated, along with its relation to non-trivial LF vacuum structure and chiral symmetry of a fermionic model.

I. INTRODUCTION

The light front (LF) form of field theory has been praised for its potential for decades (pioneered by Dirac in 1949)

essence: QFT with different choice of the space-time and field variables

Distinguished features:

- minimal number (3) of dynamical Poincaré generators
- status of the vacuum state: Fock vacuum is (almost) the true ground state (lowest-energy eigenstate of the FULL Hamiltonian) – due to positivity and conservation of the LF momentum p^+
- consistent Fock expansion of the bound states, amplitudes with direct probabilistic interpretaion á la QM

- reduction of the number dynamical field variables, constrained components (this also technically complicates the theory)

Some doubts still present: how the LF scheme can cope with the vacuum structure, condensates, symmetry breaking...

Our overall goal two-fold: test the LF scheme at the level of solvable models in D$= 1+1$ + compare with the corresponding results within the conventional (space-like, SL) theory (massless fields often involved):

try to construct non-trivial vacuum structure based on dynamical zero modes

- what is the relation between the two schemes?
  "Polyzou's paradox" - (in)equivalent representations in SL/LF form of theory. e.g.

- mass-mixing model: exact solution of both schemes including the true physical vacuum in the SL case - Bogoliubov transformation, Hamiltonian diagonalization

- can the LF theory with its drastically simplified vacuum structure generate the same predictions as the SL form?

remark: 2D massless LF fields not understood for decades

attempts essentially failed - the Schwinger model, e.g.

LF notation: $x^\mu = (x^+, x^-) = (x^0 + x^1, x^0 - x^1)$

the momentum $k^\mu = (k^+, k^-)$

$$\partial_\pm = \frac{\partial}{\partial x^\pm}, \quad \hat{k} \cdot x = \frac{1}{2} k^+ x^- + \frac{1}{2} \hat{k}^- x^+, \quad k^2 = \mu^2 \Rightarrow \hat{k}^- = \frac{\mu^2}{k^+}. \tag{1}$$

$\hat{k}^-$ is the on-shell LF energy.

No sign ambiguity analogous to $E(k^1) = \pm\sqrt{(k^1)^2 + \mu^2}$ of the conventional theory, both k^+, k^- can be taken positive

II. CHIRAL SYMMETRY - A BRIEF OVERVIEW

the simplest example - fermion field in 2D

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\,\overset{\leftrightarrow}{\partial}_\mu\,\gamma^\mu\psi - m\overline{\psi}\psi, \tag{2}$$

Gamma matrices (chiral representation):

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma^5 = \alpha^1 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Solution of $i\gamma^\mu \partial_\mu \psi(x) = m\psi(x)$:

$$\psi(x) = \int \frac{dp}{\sqrt{4\pi E(p)}} \Big[u(p) b(p) e^{-i\hat{p}\cdot x} + v(p) d^\dagger(p) e^{i\hat{p}\cdot x} \Big], \tag{4}$$

$$\hat{p}\cdot x \equiv E(p)t - px,\ \ E(p) = \sqrt{m^2 + p^2},$$

$$u(p) = \begin{pmatrix} \sqrt{p^-} \\ \sqrt{p^+} \end{pmatrix},\quad v(p) = \begin{pmatrix} -\sqrt{p^-} \\ \sqrt{p^+} \end{pmatrix},\ \ p^\pm = E(p) \pm p. \tag{5}$$

Chiral (axial) transformation:

$$\psi(x) \to e^{-i\beta\gamma^5}\psi(x) = \begin{pmatrix} e^{+i\beta}\psi_1 \\ e^{-i\beta}\psi_2 \end{pmatrix},\quad e^{-i\beta\gamma^5} = \cos(\beta)\mathbf{1} - i\sin(\beta)\gamma^5. \tag{6}$$

opposite sign in the components, chiral symmetry breaking studied in 2D models (Schwinger)

LF is different

$$\mathcal{L} = i\psi_2^\dagger \overleftrightarrow{\partial}_+ \psi_2 + i\psi_1^\dagger \overleftrightarrow{\partial}_- \psi_1 - m\big(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1\big), \psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}. \tag{7}$$

LF gamma matrices (chiral representtion):

$$\gamma^\pm = \gamma^0 \pm \gamma^1, \quad \Lambda_\pm = \frac{1}{2}\gamma^0\gamma^\pm, \quad \psi_\pm = \Lambda_\pm\psi,$$
$$\psi_+ = \begin{pmatrix}0 \\ \psi_2\end{pmatrix} = \Lambda_+\psi, \qquad \psi_- = \begin{pmatrix}\psi_1 \\ 0\end{pmatrix} = \Lambda_-\psi. \tag{8}$$

Dynamical and constrained component $(\partial_-\epsilon(x^-) = 2\delta(x^-))$

$$2i\partial_+\psi_2 = m\psi_1, \tag{9}$$
$$2i\partial_-\psi_1 = m\psi_2 \Rightarrow \psi_1(x) = \frac{m}{2i}\int dy^-\frac{1}{2}\epsilon(x^- - y^-)\psi_2(x^+, x^-). \tag{10}$$

The LF chiral transformations necessarilly

$$\psi_+(x) \to e^{-i\beta\gamma^5}\psi_2(x) \quad \text{or} \quad \psi_2(x) \to e^{-i\beta}\psi_2(x) \tag{11}$$

and **the same** transformation law for $\psi_1(x)$ due to the constraint $\Rightarrow$ no real spinor structure, no chiral symmetry $\Rightarrow$ how to obtain fermion condensate in the LF models then and what is the interpretation if no chiral SB present?

Differences translate also to $D = (3+1)$: D. Mustaki, Chiral symmetry and the constituent quark model: A null-plane point of view, hep-ph/9404206

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{12}$$

$$\psi_+(x) = \Lambda_+\psi(x) = \begin{pmatrix}\psi_1\\0\\0\\\psi_4\end{pmatrix}, \quad \psi_-(x) = \Lambda_-\psi(x) = \begin{pmatrix}0\\\psi_2\\\psi_3\\0\end{pmatrix}. \tag{13}$$

The fermionic constraint contains γ-matrices and the basic transformation

$$\psi_+(x) \to e^{-i\beta\gamma^5}\psi_+(x) = \begin{pmatrix}e^{-i\beta}\psi_1\\0\\0\\e^{-i\beta}\psi_4\end{pmatrix} \tag{14}$$

induces the law for $\psi_-(x)$

usual picture - invariance under

$$\begin{aligned}&\psi(x) \to e^{-i\beta\gamma^5}\psi(x)\\&\Rightarrow j_5^\mu(x) = \overline{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j_5^\mu = 2im\overline{\psi}\gamma^5\psi.\end{aligned} \tag{15}$$

LF case: free massive Hamiltonian is invariant

$$\psi_+(x) \to e^{-i\beta\gamma^5}\psi_+(x), \delta\psi_+ = -i\beta\gamma^5\psi_+ \Rightarrow \tag{16}$$
$$\delta\psi_-(x) = -\beta\gamma^5 \int dy^- \frac{\epsilon(x^- - y^-)}{4}\big(i\gamma^i\partial_i - m\big)\gamma^+\psi_+(x^+, y^-, x_\perp)$$

Different from conventionally obtained $j_5^\mu(x)$:

$$\delta\psi_-(x) = -i\beta\gamma^5 \int dy^- \frac{\epsilon(x^- - y^-)}{4} i\big(i\gamma^i\partial_i + m\big)\gamma^+\psi_+(x^+, y^-, x_\perp)$$
$$\tilde{j}_5^\mu(x) = j_5^\mu(x) + im\overline{\psi}\gamma^\mu\gamma^5 \int dy^- \frac{\epsilon(x^- - y^-)}{2}\gamma^+\psi_+(x^+, y^-, x_\perp) \tag{17}$$

conserved for vanishing mass

Generalization to LF QCD (3+1): Meng-Hsiu Wu, Wei-Min Zhang: Chiral symmetry in light-front QCD, JHEP 04 (2004) 045.

they derived a similar new current conserved for $m_q = 0$, chiral symmetry violated by helicity-flip interaction q-g terms

LF Chiral symmetry breaking

Definition: Hamiltonian is invariant under a group of unitary transformations, its ground state $|\Omega\rangle$ is NOT:

$$\begin{aligned} &H|\Omega\rangle = 0, \quad U(\beta) = e^{-i\beta Q}, \quad U^\dagger = U^{-1}, \\ &UHU^{-1} = H \Rightarrow [H, Q] = 0, \\ &U(\beta)HU^{-1}(\beta)U(\beta)|\Omega\rangle = 0 \Rightarrow HU(\beta)|\Omega\rangle = 0. \end{aligned} \tag{18}$$

This implies a family $U(\beta)|\Omega\rangle$ of ground states with the same energy (degeneracy) and also a non-zero vacuum expectation value of some operator (order parameter)

How this would work in the LF field theory with its kinematically defined vacuum?

First paper: C. Dietmaier, T. Heinzl, M. Schaden, and E. Werner: The Fermion Condensate of the Nambu-Jona-Lasinio Model in Light-Cone Quantization, Zeit. Phys. A 334, 215-220 (1989)

Q: how to reconcile "trivial" (empty) LF vacuum with a necessity to have a fermion condensate (vacuum expectation value of a fermion bilinear)?

Solution of the fermionic constraint in a mean-field approximation

Later followed by a series of papers by K. Itakura and Maedan

Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi + g_0\big[(\overline{\psi}\psi)^2 - (\overline{\psi}\gamma_5\psi)^2\big]. \tag{19}$$

Chiral invariant for $m_0 = 0$. Non-renormalisable, defined only with an UV cutoff. In mean field approximation, the mass m of the physical fermion determined from the gap equation

$$m - m_0 = 2ig_0 Tr S_F(0) = \frac{ig_0 m}{2(\pi)^4}\int \frac{d^4k}{k^2 - m^2 + i\epsilon} f(\Lambda, k). \tag{20}$$

The physical vacuum after a Bogoliubov transformation - a coherent state of fermion-antifermion pairs

$$|\Omega\rangle = N \exp\Big[-\sum_\lambda \int d^3k \frac{\gamma_k(m)}{\alpha_k(m)} b_0^\dagger(k,\lambda) d_0^\dagger(-k,\lambda)\Big]|0\rangle. \tag{21}$$

In the LF version, the constraint equaton is

$$2i\partial_-\psi_-(x) = \Big\{ - i\alpha^i\partial_i + \gamma^0\big[m_0 - 2g_0(\overline{\psi}\psi - \overline{\psi}\gamma_5\psi\gamma_5)\big]\Big\}\psi_+(x) \qquad (22)$$

Highly non-linear, the operator expression in $[...]$ replaced by an effective mass:

$$\begin{aligned} 2i\partial_+\psi_+(x) &= (-i\alpha^i\partial_i + \gamma^0 m)\psi_-(x), \\ 2i\partial_-\psi_-(x) &= (-i\alpha^i\partial_i + \gamma^0 m)\psi_+(x). \end{aligned} \qquad (23)$$

ψ_- eliminated and inserted to the LF Hamiltonian, part bilinear in new creation and annihilation operators (which are mass-independent at $x^+ = 0$)

leads to the equation

$$m - m_0 = \frac{g_0 m}{2\pi^3} \int\limits_0^\infty \frac{dk^+}{k^+} d^2 k_\perp f(\Lambda, k). \tag{24}$$

A series of LF papers using this approach in the large-N limit:

1. K. Itakura, Dynamical symmetry breaking in light front Gross-Neveu model, hep-th/9608062

2. K. Itakura, S. Maedan, Spontaneous symmetry breaking in discretized light cone quantiation, Prog. Theor. Phys. 97 (1997) 635-652

3. K. Itakura, Gap equations ifrom fermionic constraints on the light front, Prog. Theor. Phys. 98 (1977) 527-532

4. K. Itakura, S. Maedan, Dynamical chiral symmetry breaking on gthe light front. 1. DLCQapproach, Phys. Rev. D 61 (2000)i 045009

5. Dynamical chiral symmetry breaking on the light front. 2. The Nambu-Jona-Lasinio model, Phys. Rev. D 62 (2000) 105016

6. K. Itakura, S. Maedan, Light front realization of chiral symmetry breaking. Prog. Theor. Phys. 105 (2001) 537-571

M. Burkardt, El Khozondar, (3+1)-dimensional light-front model with spontaneous breaking of chiral symmetry, Phys. Rev. D 55 (1997) 6514-6521

Another systematic approach:

S. Beane, Broken chiral symmetry on a null plane, Annals of Physics 337 (2013), 111-142

all effects of chiral SB found in three interaction-dependent Poincaré generators (Hamiltonians), while vacuum invariant (LF Fock vacuum)

II. LF ZERO MODES - A BRIEF OVERVIEW

constrained and **dynamical** zero modes (review: M. Burkardt, Adv. Nucl. Phys. 1996)

periodic boundary conditions for finite $L, L_\perp$

$\approx$ Fourier modes with $p^+ = 0$: $\phi(x) = \phi_0(x^+, x_\perp) + \phi_N(x^+, x^-, x_\perp)$

LF scalar field

Nakanishi and Yamawaki, NPB 1977

general potential $V(\phi)$, Dirac-Bergmann quantization for constrained systems

$$\partial_\mu \partial^\mu = 4\partial_+\partial_- - \partial_\perp^2 \Rightarrow \phi_0(x^+, x_\perp) = \frac{\lambda}{\mu^2} \int\limits_{-L}^{+L} \frac{dx^-}{2L} \left(\phi_0 + \phi_n\right)^3$$

ϕ_0 is a non-linear operator function of all normal modes

LF SSB in $\lambda\phi^4(1+1)$ (Pinsky, van de Sande and Hiller 1995)

Yukawa model: G. McCartor and D. Robertson, Z. Phys. C 1992:

$$\phi_0(x^+, x_\perp) = \frac{g}{2L}(\partial_\perp^2 - \mu^2)^{-1} \int\limits_{-L}^{L} dx^- \overline{\psi}(x)\psi(x). \tag{25}$$

Recent review: S. Chabysheva and J. Hiller, Zero modes on the light front, Europhys. Journal ST (2025)

also "perturbation-theory zero modes": in Feynman diagrams rewritten in terms of LF variables, intergration from $-\infty$ to $+\infty$, end-point or delta-function contributions, not actually k^+-specific, just integration variable

DYNAMICAL ZM: independent Fourier modes, satisfy (projected) field equations with time derivatives

Examples from SL theory

Quantization of a 2D massive scalar field in a finite volume

the classical Lagrangian and field equation given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2, \quad \hat{L} = \int_{-L}^{+L} dx^1 \mathcal{L} \tag{26}$$

$$\left(\partial_\mu\partial^\mu + \mu^2\right)\phi(x) = 0. \tag{27}$$

system in a box, $-L \le x^1 \le L$ and prescribe periodic boundary conditions in x^1, $\phi(t,-L) = \phi(t,L)$, which render the momentum variable (and hence also the energy) discrete:

$$\phi(x) = \sum_{n=0,\pm1,\dots} N_n \big[a_n e^{-i\hat{p}_n\cdot x} + a_n^\dagger e^{i\hat{p}_n\cdot x}\big], \quad \hat{p}_n\cdot x = E_n(p)t - p_n x,$$

$$p_n = \frac{\pi}{L} n, n \in \mathbb{Z}, \quad E_n(p) = \sqrt{\mu^2 + p_n^2} \Rightarrow E_0 \equiv E(p_n = 0) = \mu,$$

$$\begin{aligned}\phi(x) = \phi_0(t) + \phi_N(x) = N_0\big(a_0 e^{-i\mu t} + a_0^\dagger e^{i\mu t}\big) + \\ + \sum_{n=\pm1,\dots} N_n \big[a_n e^{-iE_n t + i p_n^1 x^1} + a_n^\dagger e^{iE_n t - i p_n^1 x^1}\big]\end{aligned} \tag{28}$$

The normal-mode field $\phi_N(x)$ and the zero mode (ZM) $\phi_0(t)$ satisfy the (projected) field equations

$$(\partial_0^2 - \partial_1^2 + \mu^2)\phi_N(t, x^1) = 0, \quad (\partial_0^2 + \mu^2)\phi_0(t) = 0. \tag{29}$$

With the decomposition (28), the Lagrangian, conjugate momentum and the Hamiltonian are

$$\hat{L} = \frac{1}{2}\int_{-L}^{+L} dx^1 \left[(\partial_0\phi_N)^2 - (\partial_1\phi_N)^2 - \mu^2\phi_N^2 + (\partial_0\phi_0)^2 - \mu^2\phi_0^2\right], \tag{30}$$

$$\Pi_N = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_N} = \partial_0 \phi_N = -i \sum_{n=\pm 1,\ldots} N_n E_n \left[a_n e^{-iE_n t + ip_n^1 x^1} - a_n^\dagger e^{iE_n t - ip_n^1 x^1} \right],$$

$$\Pi_0 = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_0} = \partial_0 \phi_0 = -i\mu N_0 \big(a_0 e^{-i\mu t} - a_0^\dagger e^{i\mu t} \big), \tag{31}$$

$$H = L\big(\Pi_0^2 + \mu^2 \phi_0^2\big) + \frac{1}{2} \int_{-L}^{+L} dx^1 \left[\Pi_N^2 + (\partial_1 \phi_N)^2 + \mu^2 \phi_N^2 \right]. \tag{32}$$

Next, we assume the canonical quantization rules

$$\left[\phi_N(t, x^1), \Pi_N(t, y^1) \right] = i\delta_N(x^1 - y^1) \;\Leftrightarrow [a_m, a_n^\dagger] = \delta_{m,n}, \tag{33}$$

$$\left[\phi_0, \Pi_0 \right] = i\delta_0 = \frac{i}{2L} \Leftrightarrow \; [a_0, a_0^\dagger] = 1. \tag{34}$$

These are the natural generalization of the usual continuum rules. The finite-volume (periodic) delta-function is

$$\delta_p(x^1 - y^1) = \frac{1}{2L} + \frac{1}{2L} \sum_{n=\pm 1,\ldots} e^{-ip_n^1(x^1-y^1)} \equiv \delta_0 + \delta_N(x^1 - y^1). \qquad (35)$$

The above quantization rules fix the normalization constants to

$$N_n = \frac{1}{\sqrt{2L}} \frac{1}{\sqrt{2E_n}},\ E_0 \equiv E(p_{n=0} = 0) = \mu,\ \text{i.e. } N_0 = \frac{1}{\sqrt{2L}} \frac{1}{\sqrt{2\mu}}. \qquad (36)$$

Thus

$$\phi_0(t) = \frac{1}{\sqrt{4\mu L}} \big(a_0 e^{-i\mu t} + a_0^\dagger e^{i\mu t}\big),\ \Pi_0(t) = -i\sqrt{\frac{\mu}{4L}} \big(a_0 e^{-i\mu t} - a_0^\dagger e^{i\mu t}\big). \qquad (37)$$

After normal ordering, the zero-mode part of the Hamiltonian has the Fock form

$$H_0 = \mu a_0^\dagger a_0, \tag{38}$$

as actually expected, since there is just one dynamical zero mode with energy $E(p^1 = 0) = \mu$.

Zero-momentum modes in ideal Bose gas

an example from V. Miransky, Dynamical symmetry breaking in quantum field theories

$$\mathcal{L} = i\varphi^\dagger \frac{\partial}{\partial t}\varphi - \frac{1}{2m}\frac{\partial \varphi^\dagger}{\partial x^i}\frac{\partial \varphi}{\partial x^i} \tag{39}$$

Schroedinger equation

$$i\frac{\partial}{\partial t}\varphi + \frac{1}{2m}\frac{\partial^2 \varphi}{\partial x^i \partial x^i} = 0 \tag{40}$$

solution

$$\varphi(t,\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}} e^{-(iE(k)t - i\mathbf{k}\cdot\mathbf{x})}, \quad E(k) = \frac{k^2}{2m}, \quad [a_{\mathbf{k}}, a_{\mathbf{l}}^\dagger] = \delta_{\mathbf{kl}}.$$

Hamiltonian and number operator (N is fixed)

$$H = \sum_{\mathbf{k}} \frac{k^2}{2m} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}, \quad N = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{41}$$

ground state (vacuum) satisfies $\hat{N}|N\rangle = N|N\rangle$,

$$|N\rangle = \frac{(a_0^\dagger)^N}{\sqrt{N!}}|0\rangle, \quad a_{\mathbf{k}}|0\rangle = 0, \quad a_0^\dagger \equiv a_{\mathbf{k}=0}. \tag{42}$$

infinite volume limit, density $n = n/V$ relevant symmetry breaking related to particle number conservation $\varphi \to e^{-i\theta}\varphi$

consider the coherent state $|\theta\rangle_0, H|\theta\rangle_0 = 0$

$$|\theta\rangle_0 = \exp\big(-N/2\big)\exp(\sqrt{N}e^{i\theta}a_0^\dagger)|0\rangle,$$
$$|\theta\rangle \to U(\varphi)|\theta\rangle_0 = |\theta+\varphi\rangle_0, \quad U(\varphi) = e^{i\varphi\hat{N}}. \tag{43}$$

These coherent states are eigenstates of a_0, but not of $\hat{N}$

$$a_0|\theta\rangle_0 = \sqrt{N}e^{i\theta}|\theta\rangle_0 \to {}_0\langle\theta|a_0|\theta\rangle_0 = \sqrt{N}e^{i\theta}. \tag{44}$$

SSB: Hamiltonian invariant, vacua not

choice of the CS vacuum dictated by clusterization property **LF dynamical ZMs** in gauge and fermion fields

QED(3+1) in the (modified) LC gauge $A^+_N(x) = 0$, proper ZMs $a^\mu(x^+, x_\perp)$ constrained, $A^+_0(x^+)$ satisfies dynamical equation

$\partial^2_+ A^+_0 = eJ^-_0$

usually: A^+_0 intepreted as QM variable, "vacuum potential" (non-abelian models, A. Kalloniatis, PRD 1996, e.g.)

Relation to the second-quantized picture where (naively)

$p^- = \frac{p^2_\perp + m^2}{p^+} = \infty$?

$A^+(x^+)$ is just one mode, but what is its LF energy ? $(0/0)$

also: J^-_0 is a complicated composite operator which contains normal modes of diffent fields, treated here in a very simplified form as an external source, the pure-glue model has currents made from the gauge fields only - an advantage

ZERO MODES in an abelian gauge theory

A.C. Kalloniatis H.C. Pauli, Z. Phys. C 60, 255 (1993)
A.C. Kalloniatis, H.C. Pauli, Z. Phys. C 63, 161 (1994)
A.C. Kalloniatis, D.G. Robertson, Phys. Rev. D 50, 5262 (1994)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi - eJ_{\mu}A^{\mu} \tag{45}$$

Field equations

$$i\gamma^\mu \partial_\mu \psi - m\psi = e\gamma_\mu A^\mu \psi, \tag{46}$$

$$\partial_\mu F^{\mu\nu} = eJ^\nu, \qquad J^\mu = \overline{\psi}\gamma^\mu \psi, \tag{47}$$

or

$$(2i\partial_+ - eA^+)\psi_+ = -i\big[\alpha^i \partial_i + m\beta + e\alpha_i A^i\big]\psi_- \tag{48}$$

$$(2i\partial_- - eA^-)\psi_- = -i\big[\alpha^i \partial_i + m\beta + e\alpha_i A^i\big]\psi_+ \tag{49}$$

$$(4\partial_+\partial_- - \partial_\perp^2)A^\mu - \partial^\mu(\partial_+ A^+ + \partial_- A^- + \partial_i A^i) = J^\mu \tag{50}$$

box with $-L \leq x^- \leq L, \quad -L_\perp \leq x^i \leq L_\perp$, periodic BC except $\psi(x^+, x^-, x_\perp)$ - antiperiodic in x^-

Normal mode fields $A(x^+, x^-, x_\perp)$, **proper** zero mode field $a(x^+, x_\perp)$, **global** ZM field $A_0(x^+)$

projection of Maxwell eqs. onto the corresponding sector

$$2\partial_+\partial_- A_N^+ - (\partial_-)^2 A_N^- - 2\partial_-\partial_j A_N^j - \partial_\perp^2 A_N^+ = J_N^+ \quad (51)$$

$$2\partial_+\partial_- A_N^- - (\partial_+)^2 A_N^+ - 2\partial_+\partial_j A_N^j - \partial_\perp^2 A_N^- = J_N^- \quad (52)$$

$$(4\partial_+\partial_- - \partial_\perp^2) A_N^i + \partial_i\partial_+ A_N^+ + \partial_i\partial_- A_N^- + \partial_i\partial_j A_N^j = J_N^i \quad (53)$$

Gauge choice $A_N^+ = 0$ or $\partial_- A^+ = 0$

Gauge transformation

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\Lambda(x) \Rightarrow A^+ \to A^+ - 2\partial_-\Lambda(x). \quad (54)$$

thus x^--independent components are gauge-invariant proper ZM sector (no x^--dependence, $x_\perp$ kept) - omit ∂_- terms

$$-\partial_\perp^2 a^+ = ej^+ \tag{55}$$

$$-\partial_\perp^2 a^- - 2(\partial_+)^2 a^+ - 2\partial_+\partial_i a^i = ej^- \tag{56}$$

$$-\partial_\perp^2 a^i + \partial_i\partial_+ a^+ + \partial_i\partial_j a^j = ej^i. \tag{57}$$

the gauge condition $\partial_i a^i = 0$ (or $\partial_i a^i = -e(1-\alpha)\frac{1}{\partial_\perp^2}\partial_i j^i$)

$$a^\mu = -e\frac{1}{\partial_\perp^2} j^\mu. \tag{58}$$

All proper ZM components constrained, equations non-linear however, currents $j^\mu(x^+, x_\perp)$ contain other fields, perturbative treatment to derive Poincaré generators

global ZM sector - omit ∂_-, ∂_i terms

$J_0^+ = 0$ is first-class constraint, only as condition on states

$$J_0^+|phys\rangle = 0. \tag{59}$$

most important

$$-2\partial_+^2 A_0^+ = eJ_0^-. \tag{60}$$

LF Hamiltonian in the global zero-mode sector ($\Pi_0^- = \partial_+ A_0^+$)

$$P_G^- = V\left[\frac{1}{2}(\Pi_0^-)^2 + eA_0^+ J_0^-\right]. \tag{61}$$

In the lowest-order perturbation theory, the Hilbert space given as $\Psi \otimes |phys\rangle$

$\Psi =$ stationary wavefunctions satisfying the Schroedinger eq.

$$-\frac{1}{2}\frac{d^2}{dq^2}\Psi = \frac{e}{V}qv\Psi = \varepsilon\Psi, \tag{62}$$

where

$$q = VA_0^+, \quad v = J_0^-, \quad \varepsilon = E/V. \tag{63}$$

First free "particle", wavefunction is

$$\Psi^{(0)}(q) = \mathcal{A}e^{i\sqrt{2\varepsilon}q} \tag{64}$$

gauge freedom left: residual (large) gauge transformations maintaining the gauge condition $\partial_\mu A^+ = 0$, corresponding to the gauge function $\Lambda_0 = \frac{\pi}{eL}nx^-$:

$$A_0^+ \to A_0^+ - \frac{2\pi}{eL}n, \quad q \to q - \frac{2\pi n}{eL}V. \tag{65}$$

Boundary condition (Manton)

$$\Psi^{(0)}(q) = \Psi^{(0)}(q - \frac{2\pi n}{eL}) \tag{66}$$

Taking $n = 1$ ("first horizon") determines the wavefunction and energy:

$$\Psi_m^{(0)}(q) = \mathcal{A}e^{i\sqrt{2\varepsilon_m}q}, \quad \varepsilon_m = \frac{e^2m^2}{8(2L_\perp)^2}. \tag{67}$$

Transitions between m-states suppressed for finite $L_\perp$, their overlap vanishes, background effect, can be ignored

GENERALIZATIONS:

A.C. Kalloniatis. H.C. Pauli, S.S. Pinsky: "Dynamical zero modes and pure glue QCD in $1 + 1$ dimensions in light-cone field theory", Phys. Rev. D 50, 6633 (1994)

only ZM sector, complicated potential, lowest-energy vacuum wave function

S.S Pinsky, A.C. Kalloniatis: Light front QCD$(1+1)$ coupled to adjoint scalar matter", Phys. Lett. B 365 225 (1996)

A.S. Mueller, A.C. Kalloniatis, H.C. Pauli:"Effect of zero modes on the bound state spectrum in light cone quantization", Phys. Lett. B 435, 189 (1998)

A fresh look at LF zero modes

Generally, it was accepted (Burkardt) that from the field-theoretical point of view the dynamical LF zero modes have infinite LF energy for a massive field, although their dispersion relation $p^- = (p_\perp^2 + m^2)/p^+$ is actually ill defined $(0/0)$ for $m = 0$.

Novel formulation of the dynamical LF zero modes:

consistency of certain two-point functions of massless fields in the sector with vanishing perpendicular momentum requires presence of the ZM fields they satisfy the corresponding projection of field equations

quanta of these fields can give non-trivial Fock structure to the LF vacuum state

quantum fields, not single modes, second-quantized description

gauge-zero modes in LF QED(3+1): finite volume, periodic fields, covariant (Feynman) gauge

$$
\begin{aligned}
&4\partial_+\partial_-\tilde{A}_0^\mu(x^+,x^-) = eJ_0^\mu(x^+,x^-), \\
&A_0^\mu(x^+)
\end{aligned}
\tag{68}
$$

free field plus interaction term, zero-mode components for all μ

Advantage: can be quantized as a free field, scalar-field case useful

Also: fermionic constraint can contain a (massless) ZM field component

LF scalar field in two dimensions

essential features of our formulation present already at the level of a scalar field

begin with the 2D case, discussion will follow the continuum treatment

Lagrangian and the field equation in terms of the LF variables

$$\mathcal{L} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2, \quad \left(4\partial_+\partial_- + \mu^2\right)\phi(x) = 0. \tag{69}$$

The quantum solution of the field equation (69) is

$$\phi(x) = \sum_n \frac{1}{\sqrt{2Lk_n^+}}\Big[a_n e^{-i\hat{k}_n\cdot x} + a_n^\dagger e^{i\hat{k}_n\cdot x}\Big], \tag{70}$$

$$\hat{k}_n \cdot x \equiv \frac{1}{2}\big(k_n^+ x^- + \frac{\mu^2}{k_n^+}x^+\big).$$

The discretized momenta and the Fock commutators are

$$k_n^+ = \frac{2\pi}{L}n, \quad \big[a_m, a_n^\dagger\big] = \delta_{mn}, \quad \big[a_m, a_n\big] = 0. \tag{71}$$

The index n runs over integers for periodic boundary condition and over half-integers for antiperiodic BC

mode $n = 0$ actually not present due to the ZM-sector field equation $\mu^2\phi_0 = 0.$

Fock vacuum defined by $a_n|0\rangle = 0$. The conjugate momentum $\pi(x) = 2\partial_-\phi(x)$ given by the space (not time) derivative of the field. Together with $\theta(x) \equiv 2\partial_+\phi(x)$ expressed as

$$\pi(x) = -i\sum_n \frac{k_n^+}{\sqrt{2Lk_n^+}}\Big[a_n e^{-i\hat{k}_n\cdot x} - a_n^\dagger e^{i\hat{k}_n\cdot x}\Big], \tag{72}$$

$$\theta(x) = -i\sum_n \frac{\mu^2}{k_n^+}\frac{1}{\sqrt{2Lk_n^+}}\Big[a_n e^{-i\hat{k}_n\cdot x} - a_n^\dagger e^{i\hat{k}_n\cdot x}\Big]. \tag{73}$$

consider the correlation functions

$$iD_1^{(+)}(z) = \langle 0|\phi(x)\pi(y)|0\rangle = \frac{i}{2L}\sum_n e^{-\frac{i}{2}k_n^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k_n^+}(z^+ - i\epsilon^+)}, \tag{74}$$

$$iD_2^{(+)}(z) = \langle 0|\phi(x)\theta(y)|0\rangle = \frac{i}{2L}\sum_n \frac{\mu^2}{k_n^{+2}} e^{-\frac{i}{2}k_n^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k_n^+}(z^+ - i\epsilon^+)}, \quad (75)$$

where $z = x - y$. They are simply the discretized versions of the continuum expressions

$$D_1^{(+)}(z) = \int_0^\infty \frac{dk^+}{4\pi} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\epsilon^+)}, \quad (76)$$

$$D_2^{(+)}(z) = \int_0^\infty \frac{dk^+}{4\pi} \frac{\mu^2}{k^{+2}} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\epsilon^+)}. \quad (77)$$

change of variables $k^+ \to k^- = \mu^2/k^+$, the latter function is

$$D_2^{(+)}(z) = \int_0^\infty \frac{dk^-}{4\pi} e^{-\frac{i}{2}k^-(z^+ - i\epsilon^+) - \frac{i}{2}\frac{\mu^2}{k^-}(z^- - i\epsilon^-)}. \qquad (78)$$

Discretizing the representation (78) so that $k_n^- = 2\pi n/L$, obtain an alternative form of (75)

$$D_2^{(+)}(z) = \frac{1}{2L} \sum_n e^{-\frac{i}{2}k_n^-(z^+ - i\epsilon^+) - \frac{i}{2}\frac{\mu^2}{k_n^-}(z^- - i\epsilon^-)}. \qquad (79)$$

We see that $D_2^{(+)}(z) = D_1^{(+)}(z^+ \leftrightarrow z^-)$.

Setting $\mu = 0$ in the infinite sums (74),(79) - geometric series, summations can be performed explicitly:

$$D_1^{(+)}(z;\mu=0) = \frac{1}{2\pi}\frac{1}{x^- - y^- - i\epsilon^-}, \tag{80}$$

$$D_2^{(+)}(z;\mu=0) = \frac{1}{2\pi}\frac{1}{x^+ - y^+ - i\epsilon^+}, \tag{81}$$

coincide with the corresponding continuum representations, L-dependence dropped out. The small imaginary terms in the discretized representation required for their convergence

The two-point functions (80),(81) $\Rightarrow$ there must exist massless versions of $\phi(x), \pi(x), \theta(x)$, reproducing these functions

Indeed, the general solution of the massless LF Klein-Gordon equation

$\partial_+\partial_-\phi_0(x) = 0$ has the form

$$\phi_0(x) = \varphi_0(x^+) + \phi_0(x^-). \tag{82}$$

Setting $\mu = 0$ in the massive solution (70) directly yields the massless field $\phi_0(x^-)$:

$$\phi_0(x^-) = \sum_n \frac{1}{\sqrt{2Lk_n^+}}\left[a_n e^{-\frac{i}{2}k_n^+ x^-} + a_n^\dagger e^{\frac{i}{2}k_n^+ x^-}\right] \tag{83}$$

with the same Fock commutators (71). The second piece $\varphi_0(x^+)$ recovered from the massive field

$$\phi(x) = \int\limits_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}}\left[a(k^+)e^{-i\hat{k}\cdot x} + a^\dagger(k^+)e^{i\hat{k}\cdot x}\right] \tag{84}$$

in the continuum representation, which upon $k^+ \to k^- = \mu^2/k^+$ takes for

$\mu = 0$ the form

$$\varphi_0(x^+) = \int_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} \left[\tilde{a}(k^-) e^{-\frac{i}{2}k^- x^+} + \tilde{a}^\dagger(k^-) e^{\frac{i}{2}k^- x^+}\right]$$

with the Fock commutators

$$\left[\tilde{a}(k^-), \tilde{a}^\dagger(l^-)\right] = \delta(k^- - l^-), \quad \left[a(k^+), \tilde{a}^\dagger(l^-)\right] = 0. \tag{85}$$

Its discretized version

$$\varphi_0(x^+) = \sum_n \frac{1}{\sqrt{2Lk_n^-}} \left[\tilde{a}_n e^{-\frac{i}{2}k_n^- x^+} + \tilde{a}_n^\dagger e^{\frac{i}{2}k_n^- x^+}\right] \tag{86}$$

denote $\theta_0 \equiv 2\partial_+ \varphi_0$.

The two-point function

$$\langle 0|\varphi_0(x^+)\theta_0(y^+)|0\rangle = \frac{i}{2L}\sum_n e^{-\frac{i}{2}k_n^-(x^+-y^+)} \tag{87}$$

agrees with the formula (79) taken for $\mu = 0$, as it should.

Being x^--independent, $\varphi_0(x^+)$ actually a *zero-mode field*. Contrary to the standard picture established in previous treatments (Kalloniatis, Pauli, Robertson, Pinski ...) not just a single mode with $k^+ = 0$, but an infinite set of modes. Any state built from the Fock operators $\tilde{a}_n^\dagger$ has finite LF energy but zero LF momentum p^+ due to $[a_m, \tilde{a}_n^\dagger] = 0$:

$$P_0^+ \tilde{a}_m^\dagger \tilde{a}_n^\dagger \ldots |0\rangle = 0,$$

$$P_0^+ = 2 \int_{-\infty}^{+\infty} dx^- (\partial_- \phi_0(x^-))^2 = \sum_n k_n^+ a_n^\dagger a_n. \tag{88}$$

possibility of a non-trivial Fock structure of the LF ground states

LF scalar field in four dimensions

A similar analysis for the 4-dimensional scalar field

The discrete representation

$$\phi(x) = \sum_{\substack{n \\ n_i = 0, \pm 1, \ldots}} \frac{1}{\sqrt{V k_n^+}} \Big[a(k_{\underline{n}}) e^{-\frac{i}{2} k_n^+ x^- - \frac{i}{2} \hat{k}_{\underline{n}}^- x^+ + i k_{n_i} x^i}$$

$$+ a^\dagger(k_{\underline{n}}) e^{\frac{i}{2} k_n^+ x^- + \frac{i}{2} \hat{k}_{\underline{n}}^- x^+ - i k_{n_i} x^i} \Big],$$

$$\hat{k}_{\underline{n}}^- = \frac{k_{n_1}^2 + k_{n_2}^2 + \mu^2}{k_n^+}, \quad V = 8 L L_\perp^2, \tag{89}$$

is the solution of the LF Klein-Gordon equation

$$\big(4 \partial_+ \partial_- - \partial_1^2 - \partial_2^2 + \mu^2 \big) \phi(x) = 0. \tag{90}$$

The two-point function $iD_1^{(+)}(z) = \langle 0|\phi(x)\pi(y)|0\rangle$ can be written

$$D_1^{(+)}(z) = D_0(z^+, z^-) + \frac{1}{V}\sum_{\substack{n \\ n_1, n_2}}{}' e^{-\frac{i}{2}k_n^+ z^- - \frac{i}{2}\hat{k}_{\underline{n}}^- z^+ + ik_{n_i} z^i},$$

$$D_0(z^+, z^-) = \frac{1}{V}\sum_n e^{-\frac{i}{2}k_n^+ z^- - \frac{i}{2}\frac{\mu^2}{k_n^+}z^+}, \tag{91}$$

component with vanishing perpendicular modes ($n_1 = n_2 = 0$) separated out

symbol $\sum'$ denotes the collection of field modes with at least one of n_1 and n_2 non-zero (and equal to $\pm 1, \pm 2, \ldots$), term $D_0(z^+, z^-)$ corresponds to the component $\phi(x^+, x^-) \equiv \phi(x; k_\perp = 0)$ of the field (89) with $n_1 =$

$n_2 = 0$,

$$\phi(x^+, x^-) = \sum_n \frac{1}{\sqrt{V k_n^+}} \Big[a(k_n^+, 0) e^{-\frac{i}{2} k_n^+ x^- - \frac{i}{2} \frac{\mu^2}{k_n^+} x^+} + a^\dagger(k_n^+, 0) e^{\frac{i}{2} k_n^+ x^- + \frac{i}{2} \frac{\mu^2}{k_n^+} x^+} \Big], \tag{92}$$

which satisfies the "reduced" equation (90)

$$(4\partial_+\partial_- + \mu^2)\phi(x^+, x^-) = 0. \tag{93}$$

steps like in the 2D case, one finds for $D_2^{(+)}(z) = -i\langle 0|\phi(x)\theta(y)|0\rangle$

$$D_2^{(+)}(z) = \tilde{D}_0(z^+, z^-) + \frac{1}{V} \sum_{\substack{n \\ n_1, n_2}}{}' e^{-\frac{i}{2}k_n^- z^+ - \frac{i}{2}\hat{k}_{\underline{n}}^+ z^- + ik_{n_i} z^i},$$

$$\hat{k}_{\underline{n}}^+ = \frac{k_{n_1}^2 + k_{n_2}^2 + \mu^2}{k_n^-}, \tag{94}$$

$$\tilde{D}_0(z^+, z^-) = \frac{1}{V} \sum_n e^{-\frac{i}{2}k_n^- z^+ - \frac{i}{2}\frac{\mu^2}{k_n^-} z^-}. \tag{95}$$

For $\mu = 0$, $\tilde{D}_0(z^+, z^-)$ becomes the zero-mode term $\tilde{D}_0(z^+)$.

indication that there must exist a field, whose correlation function reproduces this term.

In a complete analogy with the two-dimensional case, $\varphi_0(x^+)$ of the form

$$\varphi_0(x^+) = \sum_n \frac{1}{\sqrt{Vk_n^-}}\Big[\tilde{a}(k_n^-)e^{-\frac{i}{2}k_n^- x^+} + \tilde{a}^\dagger(k_n^-)e^{\frac{i}{2}k_n^- x^+}\Big],$$

$$\big[\tilde{a}(k_m^-), \tilde{a}^\dagger(k_n^-)\big] = \delta_{m,n}, \quad \big[\tilde{a}(k_m^-), a^\dagger(k_n^-)\big] = 0 \tag{96}$$

is that quantity

This zero-mode field replaces the usual single zero-mode with $k^+ = 0$ whose LF energy is infinite $(\mu^2/0)$ or whose dispersion law is ill-defined for massless case $(0/0)$.

the proposed type of the dynamical LF zero modes is present for both periodic and antiperiodic boundary conditions in x^- variable

the zero-mode field $\phi_0(x) = \phi_0(x^-) + \varphi_0(x^+)$ is the solution of the field equation (93) for $\mu = 0$ and yields the correct form of the correlation functions

$$\langle 0|\phi_0(x)\pi_0(y)|0\rangle = \frac{i}{V}\sum_n e^{-\frac{i}{2}k_n^+(x^- - y^-)}$$
$$= \frac{1}{4L_\perp^2}\frac{1}{2\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \quad (97)$$
$$\langle 0|\phi_0(x)\theta_0(y)|0\rangle = \frac{i}{V}\sum_n e^{-\frac{i}{2}k_n^+(x^+ - y^+)}$$
$$= \frac{1}{4L_\perp^2}\frac{1}{2\pi}\frac{1}{(x^+ - y^+ - i\epsilon^+)}. \quad (98)$$

LF fermion field

approach extended to the fermion field in 4D

chiral representation (Kogut and Soper) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ diagonal

Dirac's γ-matrices γ^μ define also the LF matrices $\gamma^\pm = \gamma^0 \pm \gamma^3$

projection operators $\Lambda_\pm = 1/4\gamma^\mp\gamma^\pm$ and the matrices $\alpha^i = \gamma^0\gamma^i, i = 1, 2.$

Using the projectors $\Lambda_+ = \mathrm{diag}(1\ 0\ 0\ 1)$ and $\Lambda_- = \mathrm{diag}(0\ 1\ 1\ 0)$, the Dirac equation

$i\gamma^\mu\partial_\mu\psi(x) = m\psi(x)$

decomposed into the dynamical and constrained equations,

$$2i\partial_+\psi_+(x) = \left(m\gamma^0 - i\alpha^i\partial_i\right)\psi_-(x), \tag{99}$$

$$2i\partial_-\psi_-(x) = \left(m\gamma^0 - i\alpha^i\partial_i\right)\psi_+(x),, \tag{100}$$

where

$$\psi_+^T = (\Lambda_+\psi)^T = (\psi_1 \ 0 \ 0 \ \psi_4),$$
$$\psi_-^T = (\Lambda_-\psi)^T = (0 \ \psi_2 \ \psi_3 \ 0) \tag{101}$$

T means transposition

solution of the dynamical equation

$$\psi_+(x) = \int_{-\infty}^{+\infty} \frac{d^2p_\perp}{2\pi} \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi}} \sum_s \chi(s) \left[b(\underline{p}, s) e^{-i\hat{p}\cdot x} + d^\dagger(\underline{p}, -s) e^{i\hat{p}\cdot x} \right], \tag{102}$$

where $\chi^T(1/2) = (1 \ 0 \ 0 \ 0), \chi^T(-1/2) = (0 \ 0 \ 0 \ 1)$

the constraint equation solved by

$$\psi_-(x) = \int dy^- \frac{1}{4i} \epsilon(x^- - y^-) \left[m\gamma^0 - i\alpha^i \partial_i \right] \psi_+(x^+, y^-, x^i). \qquad (103)$$

Fock representation

$$\psi_-(x) = \int_{-\infty}^{+\infty} \frac{d^2 p_\perp}{2\pi} \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi} p^+} \sum_s \left[\tilde{u}(\underline{p}, s) b(\underline{p}, s) e^{-i\hat{p}\cdot x} + \tilde{v}(\underline{p}, s) d^\dagger(\underline{p}, s) e^{i\hat{p}\cdot x} \right], \qquad (104)$$

with the spinors

$$\tilde{u}(\underline{p}, 1/2) = \begin{pmatrix} 0 \\ p^1 + ip^2 \\ m \\ 0 \end{pmatrix}, \quad \tilde{u}(\underline{p}, -1/2) = \begin{pmatrix} 0 \\ m \\ -p^1 + ip^2 \\ 0 \end{pmatrix}$$

$$\tilde{v}(\underline{p}, 1/2) = \begin{pmatrix} 0 \\ -m \\ -p^1 + ip^2 \\ 0 \end{pmatrix}, \quad \tilde{v}(\underline{p}, -1/2) = \begin{pmatrix} 0 \\ p^1 + ip^2 \\ -m \\ 0. \end{pmatrix}$$

non-zero components of the two-point function $\langle 0|\psi_+(x)\psi_+^\dagger(y)|0\rangle_{\alpha\beta}$

correspond to $\alpha = \beta = 1$ or 4

$$S_+(z) = \int\limits_{-\infty}^{+\infty} \frac{d^2 p_\perp}{(2\pi)^2} \int\limits_{0}^{+\infty} \frac{dp^+}{4\pi} e^{-\frac{i}{2}p^+ z^- - \frac{i}{2}\hat{p}^- z^+ - ip^i z^i}, \qquad (105)$$

while the non-zero components of $\langle 0|\psi_-(x)\psi_-^\dagger(y)|0\rangle_{\alpha\beta}$ correspond to $\alpha = \beta = 2$ or 3

$$S_-(z) = \int\limits_{-\infty}^{+\infty} \frac{d^2 p_\perp}{(2\pi)^2} \int\limits_{0}^{+\infty} \frac{dp^+}{4\pi p^+} \hat{p}^- e^{-\frac{i}{2}p^+ z^- - \frac{i}{2}\hat{p}^- z^+ - ip^i z^i}. \qquad (106)$$

Here $\hat{p}^- = (p_\perp^2 + m^2)/p^+,\ p_\perp^2 = p_1^2 + p_2^2$.

change of variables $p^+ \to \hat{p}^-$ transforms the latter function to the same form as (105) but with z^+ and z^- interchanged:

$$S_-(z) = \int_{-\infty}^{+\infty} \frac{d^2p_\perp}{(2\pi)^2} \int_0^{+\infty} \frac{dp^-}{4\pi} e^{-\frac{i}{2}p^- z^+ - \frac{i}{2}\hat{p}^- z^- - ip^i z^i}. \tag{107}$$

both components of the LF spinor field satisfy the Klein-Gordon equation like the scalar field, try to follow the steps of the previous section

Fock expansion of ψ_+ field in the discrete basis

$$\begin{aligned}\psi_+(x^+, \underline{x}) = \sum_{\underline{n},s} \frac{\chi(s)}{\sqrt{V}} \Big[& b(p_{\underline{n}}, s) e^{-\frac{i}{2}p_n^+ x^- - \frac{i}{2}\hat{p}_{\underline{n}}^- x^+ + ip_{n_i} x^i} \\ & + d^\dagger(p_{\underline{n}}, -s) e^{\frac{i}{2}p_n^+ x^- + \frac{i}{2}\hat{p}_{\underline{n}}^- x^+ - ip_{n_i} x^i} \Big].\end{aligned} \tag{108}$$

For zero mass, the component of $\psi_+(x)$ with $n_1 = n_2 = 0$

$$\psi_+(x^-) = \sum_{n,s} \frac{\chi(s)}{\sqrt{V}} \left[b(p_n^+, s) e^{-\frac{i}{2} p_n^+ x^-} + d^\dagger(p_n^+, -s) e^{\frac{i}{2} p_n^+ x^-} \right] \qquad (109)$$

satisfies the "reduced" equation $\partial_+ \psi_+(x) = 0$ following from (99) and implying only x^--dependent solution

unlike the scalar-field case with its $\partial_+ \partial_-$ operator, it is thus not possible to recover the zero-mode field $\psi_+(x^+)$ here

However, the zero-mode field $\psi_-(x^+)$ may exist for a massless fermion field in the sector with $n_1 = n_2 = 0$ since such a field is the solution of the Eq.(100) which reduces to $\partial_- \psi_-(x) = 0$ in this sector.

demonstrate this explicitly - write the function $S_-(x)$ (107) for $m = 0$

in the discrete form

$$S_-(z) = \frac{1}{V} \sum_{\underline{n}} e^{-\frac{i}{2} p_n^- z^+ - \frac{i}{2} \frac{p_{n_1}^2 + p_{n_2}^2}{p_n^-} z^- + i p_{n_1} z^1 + i p_{n_2} z^2} \tag{110}$$

and isolate its zero-mode piece by setting $n_1 = n_2 = 0$:

$$S_0(x^+) = \frac{1}{V} \sum_n e^{-\frac{i}{2} p_n^- x^+}. \tag{111}$$

The corresponding field $\chi_0(x^+) \equiv \psi_-(x^+)$

$$\chi_0(x^+) = \sum_{n,s} \frac{\tilde{\chi}(s)}{\sqrt{V}} \left[\tilde{b}(p_n^-, s) e^{-\frac{i}{2} p_n^- x^+} - \tilde{d}^\dagger(p_n^-, -s) e^{\frac{i}{2} p_n^- x^+} \right],$$

$$\tilde{\chi}^T(1/2) = (0\ 0\ 1\ 0),\ \tilde{\chi}^T(1/2) = (0\ 1\ 0\ 0) \tag{112}$$

reproduces the zero-mode term $S_0(x^+)$ (111) of $S_-(x)$:

$$\langle 0|\chi_0(x^+)\chi_0^\dagger(y^+)|0\rangle = \frac{1}{V}\sum_n e^{-\frac{i}{2}p_n^-(x^+-y^+)}\Lambda_-. \tag{113}$$

zero-mode field $\chi_0(x^+)$ may play a role in giving non-trivial structure to the LF vacuum in models with chiral symmetry.

example: the sigma-model with fermions. In its LF treatment (Martinovic and Vary, PRD 2001) the vector and axial-vector charges Q, Q_5 contained a zero-mode part. These fermion zero modes were however the "usual" zero modes with infinite LF energy.

consider instead the axial-vector charge

$$Q_5 = \int_V d^3\underline{x} j_5^+(x) = \int_V d^3\underline{x} \Psi_+^\dagger(x)\gamma_5\Psi_+(x) \tag{114}$$

in novel approach

Q_5 associated with the conserved axial-vector current

$j_5^\mu(x) = \overline{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)$

assume for a while that the full massless fermion field

$$\Psi_+(x) = \psi_+(x) + \psi_0(x^+). \tag{115}$$

does contain the zero-mode piece $\psi_0(x^+)$. Its structure would necessarily coincide with $\chi_0(x^+)$ (112)

the only difference: spinor $\chi(s)$ instead of $\tilde{\chi}(s)$ (and the minus sign between the two terms).

this difference probably not important for the operators related to the non-trivial vacuum structure (charges)

presence of $\psi_0(x^+)$ implies that in addition to the usual Q^5_N of the axial-vector charge

$$Q^5_N = \sum_{n,s} 2s \Big[b^\dagger(p_{\underline{n}}, s) b(p_{\underline{n}}, s) + d^\dagger(p_{\underline{n}}, s) d(p_{\underline{n}}, s) \Big] \qquad (116)$$

diagonal in creation and annihilation operators and thus annihilating the

Fock vacuum, there will be a zero-mode part of the form

$$
\begin{aligned}
Q_0^5 &= \int_V d^3\underline{x}\psi_0^\dagger(x^+)\gamma^5\psi_0(x^+) \\
&= \sum_{m,s}\chi^\dagger(s)\big(b_0^\dagger(p_m^-,s)e^{\frac{i}{2}p_m^- x^+} + d_0(p_m^-,-s)e^{-\frac{i}{2}p_m^- x^+}\big) \\
\times\gamma^5 &\sum_{n,s'}\chi(s')\big(b_0(p_n^-,s')e^{-\frac{i}{2}p_n^- x^+} + d_0^\dagger(p_n^-,-s')e^{\frac{i}{2}p_n^- x^+}\big).
\end{aligned}
$$

At $x^+ = 0$, this becomes

$$
Q_0^5 = Q_{0d}^5 + \mathcal{Q}_v^5, \tag{117}
$$

$$Q^5_{0d} = \sum_{m,n,s} 2s\left[b^\dagger_0(p^-_m,s)b_0(p^-_n,s) + d^\dagger_0(p^-_n,s)d_0(p^-_m,s)\right],$$

$$\mathcal{Q}^5_v = \sum_{m,n,s} 2s\left[b^\dagger_0(p^-_m,s)d^\dagger_0(p^-_n,-s) + d_0(p^-_m,-s)b_0(p^-_s,s)\right].$$

Calculate now the zero-mode part of the axial-vector charge from the field $\chi_0(x^+) =$ dof actually at our disposal:

$$\tilde{Q}^5_0 = \int_v d^3\underline{x}\chi^\dagger_0(x^+)\gamma^5\chi_0(x^+)_{|x^+=0} = -Q^5_{0d} + \mathcal{Q}^5_v. \tag{118}$$

vacuum terms as before, diagonal part opposite sign

The unitary operator $V(\beta)$, implementing the axial-vector transformation at the quantum level, $V(\beta) = \exp\{-i\beta Q_5\} = \exp\{-i\beta Q^5_0\}\exp\{-i\beta Q^5_N\}$, would therefore contain a zero-mode part Q^5_0 which would transform the LF

Fock vacuum to a set of more complex ground states, labeled by a the real parameter β.

terms Q^5_{0d} and $\mathcal{Q}^5_v$ commute, the operator

$$|0\rangle \to |\beta\rangle = \mathcal{N} \exp\Big\{ -i\beta \sum_{m,n,s} 2s \big[b^\dagger_0(p^-_m, s) d^\dagger_0(p^-_n, -s) + b_0(p^-_m, s) d_0(p^-_n, s) \big] \Big\} |0\rangle, \tag{119}$$

where $\mathcal{N}$ is the factor obtained after commuting the terms in (117) using the Bakker-Hausdorff theorem

Each member of this set is non-invariant under $|\beta\rangle \to V(\beta')|\beta\rangle = |\beta + \beta'\rangle$ and has vanishing LF momentum

$$P^+|\beta\rangle = 0 \tag{120}$$

If the LF Hamiltonian of the model is invariant under the axial-vector transformations (always the case for massless fermion fields) – a simple realization of chiral symmtery breaking in LF field theory

LF sigma model with fermions - role of the ZM field

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \Psi - m\overline{\Psi}\Psi + \frac{1}{2}\big(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi\partial^\mu\pi\big) \\ -\frac{1}{2}\mu^2\big(\sigma^2 + \pi^2\big) - g\overline{\Psi}\big(\sigma + i\gamma^5\pi\big)\Psi \tag{121}$$

invariant with respect to the axial-vector transformations

$$\Psi(x) \to \exp\big(-i\beta\gamma^5\big)\Psi(x), \tag{122}$$

$$\sigma(x) \to \sigma(x)\cos 2\beta - \pi\sin 2\beta,$$

$$\pi(x) \to \sigma(x)\sin 2\beta + \pi(x)\cos 2\beta.$$

The corresponding coupled field equations

$$i\gamma^\mu \partial_\mu \Psi(x) = m\Psi(x) + g\big(\sigma(x) + i\gamma^5 \pi(x)\big)\Psi(x), \tag{123}$$
$$\partial_\mu \partial^\mu \sigma(x) = -\mu^2 \sigma(x) - g\overline{\Psi}(x)\Psi(x), \tag{124}$$
$$\partial_\mu \partial^\mu \pi(x) = -\mu^2 \pi(x) - ig\overline{\Psi}(x)\gamma^5 \Psi(x). \tag{125}$$

enclose the system in a "box" $-L \leq x^- \leq L, -L_\perp \leq x_\perp \leq L_\perp$ and impose periodic boundary conditions for all fields except for the fermion field in x^--variable - antiperiodic ones

three constraints among the corresponding LF field equations - the usual fermionic constraint

$$2i\partial_- \Psi_1(x) = \left(m\gamma^0 - i\alpha^i \partial_i + g\gamma^0 [\sigma(x) + i\gamma^5 \pi(x)]\right)\Psi_2(x), \tag{126}$$

two zero-mode constrainst for the scalar fields

$$(\partial_\perp^2 - \mu^2)\sigma_0 = g \int\limits_{-L}^{L} \frac{dx^-}{2L} \Big({\Psi_2}^\dagger \gamma^0 \Psi_1 + {\Psi_1}^\dagger \gamma^0 \Psi_2 \Big), \qquad (127)$$

$$(\partial_\perp^2 - \mu^2)\pi_0 = g \int\limits_{-L}^{L} \frac{dx^-}{2L} \Big({\Psi_2}^\dagger \gamma^0 \gamma^5 \Psi_1 + {\Psi_1}^\dagger \gamma^0 \gamma^5 \Psi_2 \Big). \qquad (128)$$

The scalar fields in the Eq.(126) accordingly decomposed as $\sigma(x) = \sigma_N(x) + \sigma_0(x^+, x_\perp), \pi(x) = \pi_N(x) + \pi_0(x^+, x_\perp)$; σ_N and π_N are the normal-mode fields depending on all three space variables

LF Hamiltonian calculated from the field conjugate momenta has the form

$$
\begin{aligned}
P^{-} = \int_V d^3\underline{x}\Big[&(\partial_i\sigma_0)^2 + (\partial_i\pi_0)^2 + \mu^2(\sigma_0^2 + \pi_0^2) \\
&+\big[(\partial_i\sigma_N)^2 + (\partial_i\pi_N)^2 + \mu^2(\sigma_N^2 + \pi_N^2)\big] \\
&+{\Psi_2}^\dagger(m\gamma^0 - i\alpha^i\partial_i)\Psi_1 + {\Psi_1}^\dagger(m\gamma^0 - i\alpha^i\partial_i)\Psi_2 \\
&+g{\Psi_2}^\dagger\gamma^0\big((\sigma_0 + \sigma_N) + i\gamma^5(\pi_0 + \pi_N)\big)\Psi_1 \\
&+g{\Psi_1}^\dagger\gamma^0\big((\sigma_0 + \sigma_N) + i\gamma^5(\pi_0 + \pi_N)\big)\Psi_2\Big].
\end{aligned} \tag{129}
$$

The field $\Psi_1(x)$ determined from the constraint (126), whose solution contains also the zero-mode piece $\psi_0(x^+) \equiv \chi_0(x^+)$:

$$\Psi_1(x) = \chi_0(x^+) + \int_{-L}^{L} dy^- \frac{\epsilon(x^- - y^-)}{4i} \big[m\gamma^0 - i\alpha^i \partial_i \tag{130}$$

$$+ g\gamma^0(\sigma_0 + i\gamma^5 \pi_0) + g\gamma^0(\sigma_N + i\gamma^5 \pi_N) \big] \Psi_2(x^+, y^-, x_\perp).$$

A system of coupled non-linear equations whose general solution is too difficult.

For our purpose sufficient to solve the zero-mode constraints (127),(128) to the lowest-order in the coupling constant g:

$$\sigma_0 = \frac{g}{\partial_\perp^2 - \mu^2} \int_{-L}^{L} \frac{dx^-}{2L} \Big(\psi_+^\dagger \gamma^0 \tilde{\psi_-} + \tilde{\psi_-^\dagger} \gamma^0 \psi_+ \Big), \tag{131}$$

$$\pi_0 = \frac{g}{\partial_\perp^2 - \mu^2} \int_{-L}^{L} \frac{dx^-}{2L} \Big(\psi_+^\dagger \gamma^0 \gamma^5 \tilde{\psi_-} + \tilde{\psi_-^\dagger} \gamma^0 \gamma^5 \psi_+ \Big). \tag{132}$$

where ψ_+ is the free field (108), $\tilde{\psi_-} = \psi_- + \chi$ and ψ_- is the corresponding free constrained component

the zero-mode field is present also in the LF Hamiltonian

a potential impact also on other observables

In particular, we can expect non-zero fermion condensate $\langle\beta|\overline{\Psi}(0)\Psi(0)|\beta\rangle =$

Summary and Conclusions

a novel quantization scheme for dynamical light-front zero modes

based on two-dimensional massless LF fields

necessary existence of such zero-mode fields for the four-dimensional fermion (and also scalar and possibly gauge) field derives from the consistency of the two-point functions in the massless limit in the sector with vanishing perpendiculat momentum

analysis was performed in a finite volume with fields (anti)periodic in three LF space dimensions

States built from the creation operators of the zero-mode field have vanishing p^+ and contrary to previous treatments finite - not infinite or undefined - LF energy

We expect that the gauge zero-mode field having in the Feynman gauge the (generic) form

$$A_0^\mu(x^+) = \sum_n \frac{1}{\sqrt{V k_n^-}} \big[\tilde{a}^\mu(k_n^-) e^{-\frac{i}{2}k_n^- x^+} + \tilde{a}^{\mu\dagger}(k_n^-) e^{\frac{i}{2}k_n^- x^+}\big]. \tag{133}$$

may be relevant for adding vacuum structure to gauge-theory models (like the Schwinger model or perhaps also to non-abelian theories) by for example quantum realization of residual (large) gauge symmetry (L. Martinovic, PLB 2001).

Here we have shown how this type of mechanism could work in fermionic theories: unitary operators implementing axial-vector symmetry of massless fermion fields contain fermion zero-mode terms, which modify the LF Fock vacuum leading to a degenerate set of vacuum states and consequently to chiral symmetry breaking