Subtleties of light-front two-point functions

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Abstract

A careful study of the two-point functions in light-front (LF) field theory is performed at equal LF time $x^+ = y^+$ time and at **coinciding space-time points** $x = y$. We point out that in order to obtain these limit values correctly, the corresponding Fock expansions of the LF scalar field in two and four space-time dimension should contain small imaginary parts $\pm i\epsilon^\pm$ in the $x^\pm$ terms of the plane-wave factors **exponential regularization**. As a consequence, the equal-LF-time commutators are correctly recovered from the Pauli-Jordan functions and the two-point functions calculated from the scalar field restricted to the light front $x^+ = 0$ **agree with the LF-restricted two-point function**, keeping the **mass dependence** in both quantities. At the same time, occuring singularities are regularized by $\epsilon^\pm$, in particular those present in the Green's functions for the coinciding space-time points. This treatment **corrects some statements in recent literature** that it is only the "off-shell" Feynman approach

which can access the full physical information in the light-front case. We demonstrate that the **LF Hamiltonian ("on-shell") formulation actually does not fail** to correctly predict the two-point function at coinciding space-time points (the tadpole Feynman diagram). **The LF Fock methods yield consistent results** in agreement with the Feynman formulation. In the second part, a very simple **exactly solvable model of two massive scalar fields** will be suggested as a suitable example for comparison of the **non-perturbative structure** of the LF and "equal-time" forms of field theory. Diagonalization of the corresponding Hamiltonians and construction of their physical vacuum states can help to **clarify the relation between the two** forms of QFT.

INTRODUCTION

Two topics rather briefly:

1. Consistency of the LF quantization, subtleties of two-point functions

2. Exactly solvable model to compare structure of LF and SL(=ET) forms of QFT LF quantization has been sometimes suspected even to fail in certain particular aspects

the wrong - mass-independent - form of the two-point function calculated from the scalar field restricted to the light front $x^+ = 0$ found a long time ago by Nakanishi and Yamawaki (Nucl. Phys. B 1977)

this apparent problem re-emphasized recently by Polyzou

however, the correct form of the two-point function at $x^+ = 0$ obtained if the latter computed from fields taken at $x^+ \neq 0$

his contradiction has been interpreted as a serious failure indicating non-existence of even free LF theory

another alleged inconsistency described in detail in a series of papers of P. Mannheim and collaborators

among other things, the time-ordered two-point function $D_F(x)$ of the massive scalar field in the SL and LF versions of field theory, comparing the covariant (Feynman) form with the Fock (on-shell) one

while in the case of non-zero value of x^μ all formulations gave the same result, for $x^\mu = 0$ the LF Fock calculation found to produce a different, mass-independent answer, which was obviously wrong

difficulty was attributed to an inherent inability of the LF Hamiltonian form of field theory to incorporate the contribution of the circle with infinite radius the latter turned out to be important in the LF Feynman calculation (using the residue theorem) due to the $1/k^-$ dependence of the propagator as compared to the SL dependence $1/(k^0)^2$. As a consequence, the usual pole contribution obtained by contour integration in the complex k^- plane was not sufficient to yield the correct result of the tadpole amplitude $D_F(0)$. A contribution of the circle with infinite radius, not present in the Fock approach, was found to be inevitable. The conclusion of the study was that the LF Hamiltonian approach fails in this particular problem and it is only the manifestly covariant Feynman (off-shell) form that can access relevant physical information and hence gives the correct answer.

the argumentation leading to it has certain weak points

with a consistent mathematical treatment, all methods including the LF Hamiltonian scheme yield the truly correct answer in the regularized form

Light-front two-point functions at $x^+ = y^+$ and $x = y$

quantities under study - the 2-point correlation function $D^{(+)}(x-y)$ of the massive scalar field $\phi(x)$ and the commutator function $D(x-y)$:

$$iD^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle, \tag{1}$$

$$iD(x-y) = iD^{(+)}(x-y) - iD^{(+)}(y-x). \tag{2}$$

Before considering the 4-dimensional theory, we shall study the 2D case for simplicity.

Our field expansion

$$\phi(x) = \int\limits_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \Big[a(k^+) e^{-\frac{i}{2}k^+(x^- - \frac{i}{2}\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(x^+ - \frac{i}{2}\epsilon^+)} +$$

$$+ a^\dagger(k^+) e^{\frac{i}{2}k^+(x^- + \frac{i}{2}\epsilon^-) + \frac{i}{2}\frac{\mu^2}{k^+}(x^+ + \frac{i}{2}\epsilon^+)}\Big], \quad (3)$$

$$\Big[a(k^+), a^\dagger(l^+)\Big] = \delta(k^+ - l^+), \;\; \Big[a(k^+), a(l^+)\Big] = 0 \quad (4)$$

differs from the conventional one by the convergence factors $\exp(-\frac{1}{4}k^+\epsilon^-)$ and $\exp(-\frac{1}{4}\mu^2\epsilon^+/k^+)$

$$iD^{(+)}(x-y) = \int\limits_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(x^- - y^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(x^+ - y^+ - i\epsilon^+)}. \quad (5)$$

Small imaginary parts of the arguments $(x-y)^{\pm}$ necessary for the existence of the above integral (Gradshteyn and Ryzhik, e.g.), whose explicit evaluation yields for $x^+ > 0$ (often $y = 0$ henceforth)

$$
\begin{aligned}
iD^{(+)}(x) &= \theta(-x^2)\frac{1}{2\pi}K_0\big(\mu\sqrt{-(x^+ - i\epsilon^+)(x^- + i\epsilon^-)}\big) \\
&- \theta(x^2)\frac{i}{4}H_0^{(2)}\big(\mu\sqrt{(x^+ - i\epsilon^+)(x^- - i\epsilon^-)}\big), \\
H_0^{(2)}(x) &= J_0(x) - iY_0(x).
\end{aligned}
\tag{6}
$$

$H_\nu^{(2)}(x), J_\nu(x)$, $Y_\nu(x)$ and $K_\nu(x)$ are various Bessel functions. The LF restriction of the correlation function for $x^2 < 0$ is

$$
iD^{(+)}(x^+ = 0, x^-) = \frac{1}{2\pi}K_0\big(\mu\sqrt{-i\epsilon^+|x^-|}\big) \approx -\frac{\gamma_E}{2\pi} - \frac{1}{4\pi}\ln\big(-\frac{i}{4}\mu^2\epsilon^+|x^-|\big),
\tag{7}
$$

the expansion $K_0(x) \approx -\gamma_E - \ln\frac{x}{2} + O(x^2)$ for small value of x used, γ_E is the Euler-Mascheroni constant

this expression is finite for nonzero ϵ^+

and coincides with the correlation function calculated from two scalar fields restricted to the LF, whose Fock expansion is given by setting $x^+ = 0$ in the formula (3)

in the previous treatments (Yamawaki 1998), different results were obtained depending on whether one set $x^+ = 0$ in the calculated two-point function or computed this function from the fields taken at $x^+ = 0$

In particular, it is evident from (7) that starting with $\epsilon^+ = 0$ leads to an ill defined expression $\ln(0)$.

In the time-like region, the commutator function for unequal times

$x^+ > y^+$ takes the form

$$iD(x-y) = \frac{1}{4i} H_0^{(2)}\left(\mu\sqrt{(x^+ - y^+ - i\epsilon^+)(x^- - y^- - i\epsilon^-)}\right) - c.c.. \quad (8)$$

For finite $x^- - y^-$, $iD(x-y)$ at $x^+ = y^+$ reduces to the equal-time commutator relation (ETCR)

$$\left[\phi(x^+, x^-), \phi(x^+, y^-)\right] = -\frac{i}{4}\epsilon(x^- - y^-), \quad (9)$$

where $\epsilon(x) = x/|x|$ is the sign function. This follows from $(z = x - y)$

$$iD^{(+)}(0, z^- > 0) = \langle 0|\phi(x^+, x^-)\phi(x^+, y^-)|0\rangle =$$
$$= \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{1}{2}\frac{\mu^2}{k^+}\epsilon^+} = \frac{1}{4i} H_0^{(2)}(\mu\sqrt{-i\epsilon^+|z^-|})$$
$$\approx -\frac{\gamma_E}{2\pi} - \frac{1}{4\pi}\ln\left(\frac{\mu^2|z^-|}{4}\epsilon^+\right) - \frac{i}{8}, \tag{10}$$

inserted into (8) taken at $x^+ = y^+$. The result is $-i/4$. For $z^- < 0$, the complex conjugate results in (10) and (8) found. In obtaining the expression (10), the expansions $J_0(x) \approx 1 + O(x^2)$, $Y_0(x) \approx \frac{2}{\pi}\left[\gamma_E + \ln\frac{x}{2}\right]$ valid for $x \ll 1$ were used along with the relation $\ln(i) = i\pi/2$.

Introduction of $\epsilon^+ \neq 0$ again regulates the logarithmic divergence in (10) ensuring the correct value of the ETCR (9) in a consistent way

Both $\epsilon^{\pm}$ play a role in the two-point function at the coinciding points

choosing $x^2 < 0$ for simplicity

$$iD^{(+)}(0) = \langle 0|\phi(x)\phi(x)|0\rangle = \int_0^{+\infty} \frac{dk^+}{4\pi k^+} e^{-\frac{1}{2}k^+\epsilon^- - \frac{1}{2}\frac{\mu^2}{k^+}\epsilon^+} =$$

$$= \frac{1}{2\pi} K_0(\mu\sqrt{\epsilon^+\epsilon^-}) \approx -\frac{1}{2\pi}\Big[\gamma_E + 2\ln\Big(\frac{1}{4}\mu^2\epsilon^+\epsilon^-\Big)\Big]. \quad (11)$$

the same regularized form obtained by setting $x = y$ in the integral representation (5) or in the final, integrated form (6)

both $\epsilon^{\pm}$ have to be nonzero to avoid an ill-defined expression and to achieve this agreement.

relevance of $\epsilon^+ \neq 0$ in the commutator function (8) manifests itself also for large values of its argument because in that domain $D^{(+)}(x)$ is actually damped as follows from the asymptotic expansion for $x \to \infty$

$$H_0^{(2)}(x) \approx \frac{2}{\sqrt{\pi x}} \exp\big(-i(x - \frac{\pi}{4})\big), \tag{12}$$

leading to the behaviour $\sim (\epsilon^+ x^-)^{-1/4} \exp\big(-\frac{\mu}{2}\sqrt{\epsilon^+ x^-}\big)$ for each of the two terms in the limit $x^- \to \infty$

Consequently, the commutator function at $x^+ = 0$ does not reduce to the sign function for large x^- separations but is exponentially suppressed. This solves apparent contradictions discussed extensively in [**?**, **?**] and identified as a source of inconsistency of the LF field theory (violation of vacuum simplicity or non-vanishing surface terms in the LF Poincaré algebra Yamawaki, PRD 1998)

A fully parallel treatment can be given for the (3+1)-dimensional theory

Notation: $d^2k_\perp \equiv dk^1dk^2, k_\perp^2 \equiv k_1^2 + k_2^2, k_\perp \cdot x_\perp \equiv k^1x^1 + k^2x^2, \hat{k}^- \equiv (k_\perp^2 + \mu^2)/k^+$

the corresponding field expansion

$$\phi(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int_{-\infty}^{+\infty} \frac{d^2k_\perp}{2\pi} \big[a(k^+, k_\perp)e^{-\frac{i}{2}k^+(x^- - \frac{i}{2}\epsilon^-) - \frac{i}{2}\hat{k}^-(x^+ - \frac{i}{2}\epsilon^+) + ik_\perp \cdot x_\perp}$$
$$+ a^\dagger(k^+, k_\perp)e^{\frac{i}{2}k^+(x^- + \frac{i}{2}\epsilon^-) + \frac{i}{2}\hat{k}^-(x^+ + \frac{i}{2}\epsilon^+) - ik_\perp \cdot x_\perp}\big]. \quad (13)$$

The Fock creation and annihilation operators obey

$$\big[a(k^+, k_\perp), a^\dagger(l^+, l_\perp\big] = \delta(k^+ - l^+)\delta^{(2)}(k_\perp - l_\perp), \;\; \big[a(k^+, k_\perp), a(l^+, l_\perp\big] = 0, \quad (14)$$

where the notation $\delta^{(2)}(x_\perp - y_\perp) \equiv \delta(k^1 - y^1)\delta(x^2 - y^2)$ used

In the usual treatment, the Fock commutators (14) lead to

$$\left[\phi(x^+, x^-, x_\perp), \phi(x^+, y^-, y_\perp)\right] = -\frac{i}{4}\epsilon(x^- - y^-)\delta^{(2)}(x_\perp - y_\perp). \tag{15}$$

The expansion (13) again contains the regulating terms. They are required for the integral over the k^+ variable (after performing the $d^2k_\perp$ integration) in the two-point function

$$iD^{(+)}(z) = \langle 0|\phi(x)\phi(y)|0\rangle =$$

$$\int_0^\infty \frac{dk^+}{4\pi k^+} \int_{-\infty}^{+\infty} \frac{d^2k_\perp}{(2\pi)^2} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{k_\perp^2+\mu^2}{k^+}(z^+ - i\epsilon^+) + ik_\perp \cdot z_\perp}. \tag{16}$$

to be well defined (Gradhstey and Ryzhik) For $x^+ > 0$, the result is

$$iD^{(+)}(x) = \frac{i\mu\theta(x^2)}{8\pi\sqrt{x^2}} H_1^{(2)}(\mu\sqrt{x^2}) - \frac{\mu\theta(-x^2)}{4\pi^2\sqrt{-x^2}} K_1(\mu\sqrt{-x^2}), \qquad (17)$$

where $x^2 = x^+x^- - x_\perp^2$ with the $i\epsilon^\pm$ factors implicitly present. In the space-like region, in agreement with the two-dimensional case, the direct evaluation of the two-point function in terms of LF-restricted fields agrees with the value of the two-point function (17) at $x^+ = 0$:

$$iD^{(+)}(0, x^-, x_\perp) \equiv \langle 0|\phi(0, x^-, x_\perp)\phi(0, 0, 0)|0\rangle =$$
$$= \frac{\mu}{4\pi^2\sqrt{x_\perp^2 - i\epsilon^+ x^-}} K_1(\mu\sqrt{x_\perp^2 - i\epsilon^+ x^-}). \qquad (18)$$

In the previous studies, the covariant result rewritten in terms of the LF variables gave the correct expression for $x^+ = 0$ (without the $i\epsilon^+ x^-$

term, however), the direct calculation of the LF two-point function (16) reproduced this result, but the two-point function (18) calculated from the fields restricted to $x^+ = 0$ failed to yield the correct result. Indeed, without the ϵ^+ term in the field expansion (13), one finds

$$\langle 0|\phi(0,\underline{x})\phi(0,\underline{y})|0\rangle = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d^3\underline{k}}{2k^+}\theta(k^+)e^{-i(\underline{x}-\underline{y})\cdot\underline{k}}$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\frac{dk^+}{2k^+}e^{-\frac{i}{2}k^+(x^- - y^-)}\int_{-\infty}^{+\infty}\frac{d^2k_\perp}{(2\pi)^2}e^{ik_\perp\cdot(x_\perp - y_\perp)}, \qquad (19)$$

where $\underline{x} \equiv (x^-, x_\perp), \underline{k} \equiv (k^+, k_\perp)$ and the last integral is just $\delta^{(2)}(x_\perp - y_\perp)$.

The contradiction between (18) and (19) can be viewed as arising due to the lost mass dependence in (19) or as having the x^--dependence in (19) while losing it in (18) at $x^+ = 0$ since in the Lorentz invariant two-point function $x^2 = x^+x^- - x_\perp^2$. Setting $x^+ = 0$ in the regularized Fock expansion (13), instead of the representation (19) we get

$$iD^{(+)}(x-y) = \frac{1}{2\pi}\int_0^\infty \frac{dk^+}{2k^+}\int_{-\infty}^{+\infty}\frac{d^2k_\perp}{(2\pi)^2}e^{-\frac{i}{2}k^+(x^- - y^- - i\epsilon^-) + ik_\perp\cdot(x_\perp - y_\perp) - \frac{i}{2}\frac{k_\perp^2+\mu^2}{k^+}\epsilon^+} \tag{20}$$

Non-zero ϵ^+ keeps the mass dependence in the integrand and also mixes the k^+ nad $k_\perp$ integration variables so that instead of the "factorized" form (19) one obtains (18). Thus, the problem of "the order of integration and setting $x^+ = 0$ matters" (Nakanishi and Yabuki, Polyzou) has been eliminated in the present approach.

LF Green's functions

summary of the relevant aspects of the Green's function formalism

the classical differential equation corresponding to the Klein-Gordon field equation with the delta-function source

$$\left(\partial_\nu \partial^\nu + \mu^2\right) G(x-y) = -\delta^{(4)}(x-y) \tag{21}$$

has the solution

$$G(z) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot z}}{k^2 - \mu^2}, \quad z = x - y. \tag{22}$$

In terms of the LF variables, the defining equation and its solution is

$$\left(4\partial_+ \partial_- - \partial_\perp^2 + \mu^2\right) G(z^+, z^-, z_\perp) = -\delta(z^+)\delta(z^-)\delta^{(2)}(z_\perp), \tag{23}$$

$$G_F(z^+, z^-, z_\perp) = \frac{1}{2}\frac{1}{(2\pi)^4}\int\limits_{-\infty}^{+\infty} dk^+ \int\limits_{-\infty}^{+\infty} dk^- \int\limits_{-\infty}^{+\infty} d^2k_\perp \frac{e^{-\frac{i}{2}k^-z^+ - \frac{i}{2}k^+z^- + ik_\perp\cdot z_\perp}}{k^+k^- - k_\perp^2 - \mu^2 + i\epsilon}. \tag{24}$$

In analogy to the usual SL treatment, the pole at $k^2 = \mu^2$ has been shifted in such a way that the contour integration in the k^- plane yields

$$G_F(z^+ > 0, z^-, z_\perp) = D^{(+)}(z) \tag{25}$$

$$G_F(z^+ < 0, z^-, z_\perp) = D^{(+)}(-z), \tag{26}$$

which is equal to the time-ordered product

$$iD_F^{(+)}(x-y) = i\theta(x^+ - y^+)D^{(+)}(x-y) + i\theta(y^+ - x^+)D^{(+)}(y-x), \tag{27}$$

where

$$iD^{(+)}(z) = \int\limits_0^\infty \frac{dk^+}{4\pi k^+} \int\limits_{-\infty}^{+\infty} \frac{d^2k_\perp}{(2\pi)^2} e^{-\frac{i}{2}k^+z^- - \frac{i}{2}\frac{k_\perp^2+\mu^2}{k^+}z^+ + ik_\perp \cdot z_\perp}. \qquad (28)$$

This connection can be demonstrated also in another way: using in (27) the representation (28) and the theta function in the form

$$\Theta(\tau) = \frac{i}{2\pi} \int\limits_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}, \qquad (29)$$

we obtain after changing the variables as $k^- = \hat{k}^- + \omega$ from (27) the

Green's function (24)

$$D_F^{(+)}(x) = G_F(x) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot x}}{k^2 - \mu^2 + i\epsilon}. \tag{30}$$

The LF two-point function at coinciding points

a detailed analysis of subtle properties of the $D_F^{(+)}(x)$ function in both the SL and LF forms of the field theory performed in (MBL henceforth)

* P. D. Mannheim, P. Lowdon, S. J. Brodsky, Structure of light front vacuum sector diagrams, Phys. Lett. B **797**, 134916 (2019)

* P. D. Mannheim, Equivalence of light-front quantization and instant-time quantization, Phys. Rev. D **102**, 025020 (2020)

* P. D. Mannheim, P. Lowdon, S. J. Brodsky, Comparing light-front quantization with instant-time quantization, Phys. Rep. **891**, 1 (2021)

main conclusion of that work: the LF Fock approach fails to reproduce the correct form of the $G_F(x)$ function for $x^\mu = 0$ and it is only the LF Feynman ("off-shell") formulation which works correctly

HERE: we reexamine some elements of the analyses and demonstrate that the above conclusion is not valid

First, the representation (41) below considered as the correct "vacuum-sector" value of the $D_F^{(+)}$ function is actually ill-defined

there is no reason to require that the LF Fock formalism reproduces this singular integral

Second, we shall show that our regularized field expansion (13) yields the truly correct expression of the considered two-point function for coinciding points. Its value is quadratically divergent but properly regularized by the small parameters $\epsilon^{\pm}$ present in the field expansion (13). Working with the regularized Fock expansion makes the Feynman and Fock approaches to agree in the "vacuum" as well as "non-vacuum" sectors in the both SL and LF forms of the theory. It also removes the apparent discrepancy in the LF Feynman treatment between a direct calculation of $D^{(+)}(0)$ and the $x^{\mu} \to 0$ value of $D^{(+)}(x)$. This difficulty is actually another manifestation of the "ordering-of-the-calculation-matters" - problem discussed (and solved) in the previos section.

more details of the logical scheme and computational techniques

main object of the studies was the time-ordered function

$$D(x) \equiv D_F^{(+)}(x) = -i\Big[\theta(\tau)\langle 0|\phi(x)\phi(0)|0\rangle + \theta(-\tau)\langle\phi(0)\phi(x)|0\rangle\Big] \quad (31)$$

evaluated in the SL ($\tau = t$) and LF ($\tau = x^{+}$) forms of the theory, using both the (manifestly covariant) Feynman as well as the (on-shell) Fock formalism, for $x \neq 0$ and for $x = 0$, that is considering eight separate cases. Specifically, the task was to perform the integrations in the p^0 or p^- variable in the expressions

$$D(x^\mu, \mathsf{SL}) = \frac{1}{(2\pi)^4} \int dp^0 dp^1 dp^2 dp^3 \frac{e^{-i(p_0 x^0 - p^1 x^2 - p^2 x^2 - p^3 x^3)}}{p_0^2 - p_1^2 - p_2^2 - p_3^2 - \mu^2 + i\epsilon}, \quad (32)$$

$$D(x^\mu, \mathsf{LF}) = \frac{1}{2} \frac{1}{(2\pi)^4} \int dp^- dp^+ dp^1 dp^2 \frac{e^{-i(\frac{1}{2}p^- x^+ + \frac{1}{2}p^+ x^- - ip^1 x^1 - ip^2 x^2)}}{p^+ p^- - p_1^2 - p_2^2 - \mu^2 + i\epsilon}, \quad (33)$$

$$D(x^\mu = 0, \mathsf{SL}) = \frac{1}{(2\pi)^4} \int dp^0 dp^1 dp^2 dp^3 \frac{1}{p_0^2 - p_1^2 - p_2^2 - p_3^2 - \mu^2 + i\epsilon}, \quad (34)$$

$$D(x^\mu = 0, \mathsf{LF}) = \frac{1}{2} \frac{1}{(2\pi)^4} \int dp^- dp^+ dp^1 dp^2 \frac{1}{p^+ p^- - p_1^2 - p_2^2 - \mu^2 + i\epsilon} \quad (35)$$

and then to compare the results with those obtained using the Fock formalism. Applying the contour integration, in the first three cases above the contribution of the circle with infinite radius is suppressed exponentially or by power in the denominator and therefore only the pole terms contribute. As an independent check, especially in certain subtle cases, the complementary method of the exponential α-representation (αR for short), used already in the pioneering papers (Chang and Ma, T.-M. Yan) It consists in replacing the denominators in the above integrals by the integral

$$iD^{-1} = \int_0^\infty d\alpha e^{i\alpha(D+i\epsilon)}. \tag{36}$$

For non-zero x, no contradiction was identified: SL Feynman, SL Fock, LF Feynman and LF Fock formalisms gave identical result. For example, using the αR, one finds after simple manipulations

$$D(x^0 > 0, \mathsf{SL}, \alpha R) = -\frac{i}{(2\pi)^4} \int\limits_{-\infty}^{\infty} d^4 p e^{-ip\cdot x} \int\limits_0^{\infty} d\alpha e^{i\alpha(p^2 - \mu^2 + i\epsilon)} \quad (37)$$

$$= -\frac{1}{16\pi^2} \int\limits_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-ix^2/4\alpha - i\alpha\mu^2 - \alpha\epsilon} = \frac{1}{8\pi} \frac{\mu}{\sqrt{x^2}} H_1^{(2)}(\mu\sqrt{x^2}), \quad (38)$$

which agrees with the pole contribution from the contour-integration method. The explicit forms of the p^2 and $p \cdot x$ products are given in (32). The formula (38) is valid for both time-like and space-like intervals, because of the relation $2K_1(z) = \pi H_1^{(2)}(iz)$. The same result is obtained by inserting the field expansion

$$\phi(x) = \int \frac{d^3p}{2\pi\sqrt{4\pi E_p}} \left[a(\vec{p}) e^{-iE_pt + i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p}) e^{iE_pt - i\vec{p}\cdot\vec{x}} \right],$$

$$E_p = \sqrt{p_1^2 + p_2^2 + p_3^2 + \mu^2}, \tag{39}$$

to the definition (31).

In the LF case, $D(x^+ > 0, \text{LF}, \alpha R)$ analogous to (37), with p^2 and $p \cdot x$ given in (33) again yields the result (38) in agreement with the pole contribution of the contour method. Although there is only one power of p^- in the denominator of (33), closing the integration contour for $x^+ > 0$ in the lower complex p^-plane generates an exponential suppression of the contribution of the circle with infinite radius.

The LF Fock calculation for $x^\mu \neq 0$ implies inserting the LF scalar-field

Fock expansion

$$\phi(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int_{-\infty}^{+\infty} \frac{d^2k_\perp}{2\pi} \big[a(k^+, k_\perp) e^{-\frac{i}{2}k^+x^- - \frac{i}{2}\hat{k}^- x^+ + ik_\perp \cdot x_\perp}$$

$$+ a^\dagger(k^+, k_\perp) e^{\frac{i}{2}k^+x^- + \frac{i}{2}\hat{k}^- x^+ - ik_\perp \cdot x_\perp}\big]. \quad (40)$$

to (31). The result (38) is readily reproduced. Thus, for $x^\mu \neq 0$, in both the SL and LF forms of the theory, the full result for the Feynman-diagram method is given by the pole contribution and agrees with the Fock-approach result.

In the "vacuum sector" ($x^\mu = 0$), the SL Feynman approach appeared to work reliably as well. Suppression $\sim (p^0)^{-2}$ in the integrand of (34) eliminates contribution of the semi-circle at infinity. Integrating over p^0 by the contour method then leads to the pole term which is equal to the intermediate αR-formula (38) with $x^\mu = 0$:

$$\mathcal{J}(\mu) = -\frac{1}{16\pi^2}\int\limits_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha\mu^2 - \alpha\epsilon}. \tag{41}$$

The same result is obtained by inserting the Fock expansion (39) taken at $x^\mu = 0$ to $D(x^\mu = 0) = -i\langle 0|\phi(0)\phi(0)|0\rangle$. The expression (41) was considered to be the correct result which all other methods including the LF Fock calculation should reproduce.

However, in the LF case, the evaluation of the integral (35), corresponding to the "vacuum sector", turned out to be more subtle.

Though the αR-calculation reproduced the representation (41)

$$D(x^+ > 0, \alpha \mathrm{R}) = -\frac{i}{8\pi^3} \int\limits_{-\infty}^{+\infty} dp^+ \int\limits_0^{+\infty} d\alpha e^{-i\alpha\mu^2 - \alpha\epsilon} \frac{\pi}{i\alpha} \delta(2\alpha p^+) =$$

$$= -\frac{1}{16\pi^2} \int\limits_0^{\infty} \frac{d\alpha}{\alpha^2} e^{-i\alpha\mu^2 - \alpha\epsilon}, \tag{42}$$

where the first integral is the result of $(p_\perp, p^-)$-integration, the pole contribution gave a mass-independent result

$$D(x^\mu = 0, \mathrm{pole}) = -\frac{i}{16\pi^3} \int\limits_{-\infty}^{+\infty} d^2 p_\perp \int\limits_0^{+\infty} \frac{d\alpha}{\alpha}, \tag{43}$$

indicating a nonzero contribution of the circle with infinite radius. Notice that this expression reproduces the Eq.(19) taken at $\underline{x} = \underline{y}$. Nevertheless, the picture was not completely clear. First, as recognized by the authors, $D(x^\mu = 0, \mathrm{pole})$ is not the $x^\mu \to 0$ limit of $D(x^+ > 0, \mathrm{pole})$ because contrary to (43), the latter does depend on μ Similarly, the contribution of the lower circle for $x^\mu = 0$ cannot be the limit of the lower circle contribution for $x^\mu \neq 0$ because the former is zero while the latter has to be non-zero in order to replace the mass-independent contribution (43). Moreover, the contribution of the lower circle (not calculated explicitly) would have to contain also the second, mass-independent term to cancel the pole contribution (41). The solution of this cumbersome situation was found in an independent evaluation which avoided the pole contribution altogether and obtained the result $\mathcal{J}(\mu)$ (41) entirely as the circle contribution. To achieve this, the α-representation was used and the upper and lower circles were chosen in such a way that the contour did not contain any poles.

In order to better understand reasons for the difficulties of the LF Feynman case, the authors studied the problem finally without using the contour representation for the theta function, replacing it with an alternative form

$$\theta(x^+) = \int\limits_0^\infty d\beta \delta(\beta - x^+). \tag{44}$$

Starting from the α-representation and integrating over p^-, the intermediate result

$$D(x^\mu, \alpha\mathrm{R}) =$$

$$-\frac{i}{(2\pi)^3}\int d^2p_\perp \int\limits_0^\infty dp^+ e^{-\frac{i}{2}p^+x^- - ip_\perp\cdot x_\perp} \int\limits_0^\infty d\alpha e^{-i\alpha(p_\perp^2+\mu^2-i\epsilon)}\delta(\alpha p^+ - x^+)$$

$$-\frac{i}{(2\pi)^3}\int d^2p_\perp \int\limits_{-\infty}^0 dp^+ e^{-\frac{i}{2}p^+x^- - ip_\perp\cdot x_\perp} \int\limits_0^\infty d\alpha e^{-i\alpha(p_\perp^2+\mu^2-i\epsilon)}\delta(\alpha p^+ - x^+).$$

When the α-integration was performed afterwards, the standard form (27), (28) of the time-ordered function obtained (without $i\epsilon$ term in the exponential), in which the limit $x^\mu \to 0$ reproduces the incorrect result (43).

On the other hand, setting $x^\mu = 0$ in the intermediate form (45) and performing the momentum integration yielded the desired result $\mathcal{J}(\mu)$

(41). The explanation of the failure of the first procedure: mathematical inconsistency: the integrand contains a singularity $(p_\perp^2 + \mu^2)x^+/p^+$ because the point $p^+ = 0$ is included in the integration domain.

situation is analogous to that noticed for the first time by Nakanishi and Yabuki (Lett. Math. Phys. 1977): the order of setting in their case x^+ or here x^μ to zero matters.

The main trouble identified in the MBL analysis was however associated with the LF Fock expansion (40)

setting $x^\mu = 0$ one loses any mass dependence in this expansion which, when inserted to $D(x^\mu = 0) = -i\langle 0|\phi(0)\phi(0)|0\rangle$, leads to the mass-independent pole term (43). As the authors note, this is surprising because no obvious mistake was made in the above derivation. In spite of that, their conclusion was that the LF Fock expansion fails in the "vacuum sector". The crucial argument for that statement was that the LF Feynman method requires a non-zero contribution of the circle at infinity in this case (because the exponential suppression is lost for $x = 0$ and the power suppression $1/p^-$ is not sufficient), and the LF Fock calculation misses this contribution. As we show now, this conclusion is not correct.

First observation: there is a serious problem with the integral (41): although it seems to be a μ-dependent quantity, it is actually an ill-defined (infinite) object, evident already from its equality to the $x^\mu = 0$ value of the SL two-point function (using the Fock expansion (39) with $x^\mu = 0$)

$$\mathcal{J}(\mu) = -\frac{i}{(2\pi)^3} \int \frac{d^3p}{2E_p}, \tag{45}$$

which is quadratically divergent. The same conclusion follows from the observation that the integral $\mathcal{J}(\mu)$ is the $\lambda = 0$ value of the integral

$$\mathcal{J}(\mu, \lambda) = -\frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\frac{\lambda^2}{\alpha} - i\alpha\mu^2 - \alpha\epsilon} \approx \frac{1}{\lambda^2}. \tag{46}$$

This of course is nothing but the value of the function $\mu(x^2)^{-1/2} H_1^{(2)}(\mu\sqrt{x^2})$ of Eq.(38) for $x^2 \to 0$, as follows from the expansions $J_1(\epsilon) \approx \epsilon/2$ and $Y_1(\epsilon) \approx -2/\pi\epsilon$ for very small ϵ.

It is clear that to correctly formulate both the LF Feynman and Fock approaches, one needs a regularization. The most natural is provided by the regularized field expansion (13):

$$\phi(x) = \int\limits_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int\limits_{-\infty}^{+\infty} \frac{d^2k_\perp}{2\pi} \big[a(k^+,k_\perp) e^{-\frac{i}{2}k^+(x^- - \frac{i}{2}\epsilon^-) - \frac{i}{2}\hat{k}^-(x^+ - \frac{i}{2}\epsilon^+) + ik_\perp \cdot x_\perp}$$

$$+ a^\dagger(k^+,k_\perp) e^{\frac{i}{2}k^+(x^- + \frac{i}{2}\epsilon^-) + \frac{i}{2}\hat{k}^-(x^+ + \frac{i}{2}\epsilon^+) - ik_\perp \cdot x_\perp}\big]. \quad (47)$$

The corresponding two-point function is

$$iD^{(+)}(z) = \langle 0|\phi(x)\phi(y)|0\rangle =$$

$$= \int\limits_0^\infty \frac{dk^+}{4\pi k^+} \int\limits_{-\infty}^{+\infty} \frac{d^2k_\perp}{(2\pi)^2} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{k_\perp^2+\mu^2}{k^+}(z^+ - i\epsilon^+) + ik_\perp \cdot z_\perp}. \quad (48)$$

To calculate its explicit form, set $y = 0$ for simplicity and perform a change of variables

$$-\frac{i}{2}\frac{k_\perp^2}{k^+}x^+ + ik_\perp\cdot x_\perp = -i\frac{x^+}{2k^+}\Big[p_\perp^2 - \frac{k^{+2}}{x^{+2}}x_\perp^2\Big], \quad p_\perp = k_\perp - \frac{k^+}{x^+}x_\perp. \tag{49}$$

Using the integral

$$\int\limits_{-\infty}^{+\infty} d^2p_\perp e^{-\frac{i}{2}\frac{x^+}{p^+}p_\perp^2} = -\frac{2i\pi p^+}{x^+} \tag{50}$$

one finds

$$iD^{(+)}(x) = -\frac{1}{(2\pi)^3}\frac{\pi}{x^+}\int\limits_0^\infty dp^+ e^{-\frac{i}{2}p^+(x^- - i\epsilon^- - \frac{x_\perp^2}{x^+ - i\epsilon^+}) - \frac{i}{2}\frac{\mu^2}{p^+}(x^+ - i\epsilon^+)} \tag{51}$$

which in the exponential has the structure of the two-dimensional analog (5). Taking $x^+ > 0$ and using the formula

$$\int_0^\infty dx e^{-iax-ib/x} = -i\pi\sqrt{\frac{b}{a}} H_1^{(2)}(2\sqrt{ab}) \qquad (52)$$

find

$$D^{(+)}(x^+ > 0, \text{LF}) = \frac{1}{8\pi} \frac{\mu}{\sqrt{(x^+ - i\epsilon^+)(x^- - i\epsilon^-) - x_\perp^2}}$$
$$\times H_1^{(2)}\left(\mu\sqrt{(x^+ - i\epsilon^+)(x^- - i\epsilon^-) - x_\perp^2}\right). \qquad (53)$$

can safely set $x^\mu = 0$ in this expression, the result is the same as the one obtained by inserting the $x = 0$ value of the regularized field expansion

$$\phi(0) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int_{-\infty}^{+\infty} \frac{d^2 k_\perp}{2\pi} \Big[a(k^+, k_\perp) e^{-\frac{1}{4}k^+\epsilon^- - \frac{1}{4}\frac{k_\perp^2+\mu^2}{k^+}\epsilon^+}$$

$$+ a^\dagger(k^+, k_\perp) e^{-\frac{1}{2}k^+\epsilon^- - \frac{1}{4}\frac{k_\perp^2+\mu^2}{k^+}\epsilon^+} \Big]. \quad (54)$$

into $iD(x^\mu = 0) = \langle 0|\phi(0)\phi(0)|0\rangle$:

$$iD(x^\mu = 0, \text{LF}) \quad = \quad \int_0^\infty \frac{dk^+}{4\pi k^+} \int_{-\infty}^{+\infty} \frac{d^2 k_\perp}{(2\pi)^2} e^{-\frac{1}{2}k^+\epsilon^- - \frac{1}{2}\frac{k_\perp^2+\mu^2}{k^+}\epsilon^+}, \qquad (55)$$

$$iD(x^\mu = 0, \textsf{LF}) = \frac{1}{8\pi^2\epsilon^+}\int\limits_0^\infty dk^+ e^{-\frac{1}{2}k^+\epsilon^- - \frac{1}{2}\frac{\mu^2}{k^+}\epsilon^+} = \frac{1}{4\pi^2}\frac{\mu}{\sqrt{\epsilon^+\epsilon^-}}K_1(\mu\sqrt{\epsilon^+\epsilon^-})$$

$$= \frac{1}{4\pi^2}\frac{1}{\epsilon^+\epsilon^-}, \quad (56)$$

where $K_1(\epsilon) \approx 1/\epsilon$ used as well as

$$\int\limits_{-\infty}^{+\infty} d^2k_\perp e^{-\frac{1}{2}\frac{k_\perp^2\epsilon^+}{k^+}} = \frac{2\pi k^+}{\epsilon^+} \text{ and } \int\limits_0^\infty dx e^{-ax-b/x} = 2\sqrt{\frac{b}{a}}K_1(2\sqrt{ab}). \quad (57)$$

An analogous regularization required in the Feynman form (33):

$$D(x^\mu, \text{LF}) = \frac{1}{2}\frac{1}{(2\pi)^4}\int dp^- dp^+ dp^1 dp^2 \frac{e^{-\frac{i}{2}p^-(x^+ - i\epsilon^+) - \frac{i}{2}p^+(x^- - i\epsilon^-) - ip^1x^1 - ip^2x^2}}{p^+p^- - p_1^2 - p_2^2 - \mu^2 + i\epsilon}, \tag{58}$$

For $x^\mu = 0$ we obviously get

$$D(x^\mu = 0, \text{LF}) = \frac{1}{2}\frac{1}{(2\pi)^4}\int \frac{dp^+}{p^+} dp^1 dp^2 dp^- \frac{e^{-\frac{1}{2}p^-\epsilon^+ - \frac{1}{2}p^+\epsilon^-}}{p^- - \frac{p_1^2 + p_2^2 + \mu^2 - i\epsilon}{p^+}}. \tag{59}$$

Evaluation of the p^--integral by the contour method reproduces the result (56) as the pole contribution, because the circle with infinite radius is exponentially suppressed. This can be seen by writing $p^- = Re^{i\theta}$ with $-\pi \le \theta \le 0$ and taking $p^- \to \mathrm{sgn}(\pi/2 - \arg(\mathrm{p}^-))\mathrm{p}^-$ to ensure convergence for $\theta > \pi/2$. The first integral of Eq.(56) is then given by the residue obtained by inserting the pole at $p^- = (p_\perp^2 + \mu^2)/p^+$ to p^- of the exponential of the integrand of (59).

Using the α-representation, the divergent expression (41) is replaced by

$$\mathcal{J}(\mu, reg) = -\frac{1}{16\pi^2} \int\limits_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha\mu^2 - i\frac{\epsilon^+\epsilon^-}{4\alpha}} =$$

$$= \frac{1}{8\pi} \frac{\mu}{\sqrt{\epsilon^+\epsilon^-}} H_1^{(2)}(\mu\sqrt{\epsilon^+\epsilon^-}) = -\frac{1}{4\pi^2} \frac{1}{\epsilon^+\epsilon^-}. \qquad (60)$$

in agreement with (56)

We can see that the regularization greatly simplifies the situation: no ambiguities related to the circle at infinity contributions are present, the pole contribution alone is responsible for the (regularized) correct result, which is obtained by all the methods, setting $x^\mu = 0$ at any stage of the calculations

- No failure of the LF on-shell approach, Feynman and Fock methods in a complete agreement

Elementary model: two massive scalar fields with linear coupling

nicely illustrates principal structural differences between LF and SL forms

discussed in "Quantum fields" by Bogoliubov and Shirkov to illustrate limitations of canonical quantization

similarity renormalization method by Glazek (Phys. Rev. D 2013)

two dimensions for simplicity

preliminary treatment, not quite trivial if $\mu_1 \neq \mu_2$

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \mu_1^2 \varphi_1^2 + \partial_\mu \varphi_2 \partial^\mu \varphi_2 + \mu_2^2 \varphi_2^2 - 2g\varphi_1\varphi_2\right] \tag{61}$$

surprisingly complex for SL theory, very simple for LF: comparison?

Coupled field equations

$$(\partial_\mu \partial^\mu + \mu_1^2)\varphi_1(x) = -g\varphi_2(x), \quad (\partial_\mu \partial^\mu + \mu_1^2)\varphi_2(x) = -g\varphi_1(x) \tag{62}$$

solved by rotation (implemented by a unitary operator)

$$\varphi_1 = \cos\alpha\phi + \sin\alpha\chi, \quad \varphi_2 = -\sin\alpha\phi + \cos\alpha\chi \tag{63}$$

Set the coefficient at the $\phi\chi$-term equal to zero:

$$\tan\alpha = \frac{2g}{\Delta}, \quad \Delta = \mu_2^2 - \mu_1^2, \quad \alpha = \frac{1}{4i}\ln\frac{1 + i\frac{2g}{\Delta}}{1 - \frac{2g}{\Delta}}. \tag{64}$$

This also determines the coefficients at ϕ^2 and χ^2 (masses):

$$\mu^2 = \frac{1}{2}(\mu_1^2 + \mu_2)^2 - \frac{1}{2}\sqrt{g^2 + (\mu_2^2 - \mu_1^2)^2}, \quad (65)$$

$$m^2 = \frac{1}{2}(\mu_1^2 + \mu_2)^2 + \frac{1}{2}\sqrt{g^2 + (\mu_2^2 - \mu_1^2)^2}. \quad (66)$$

So the Lagrangian becomes diagonal, the corresponding Hamiltonian is

$$H = \int dx \Big[(\partial_0\phi)^2 + (\partial_1\phi)^2 + \mu^2\phi^2 + (\partial_0\chi)^2 + (\partial_1\chi)^2 + m^2\chi^2\Big]. \quad (67)$$

Quantum level: expand

$$\varphi_1(x) = \int \frac{dx}{\sqrt{4\pi E_1(k)}} \left[a(k)e^{-iE_1(k)t+ikx} + a^\dagger(k)e^{iE_1(k)t-ikx}\right], \quad (68)$$

$$\varphi_2(x) = \int \frac{dx}{\sqrt{4\pi E_2(k)}} \left[c(k)e^{-iE_2(k)t+ikx} + c^\dagger(k)e^{iE_2(k)t-ikx}\right],$$

$$E_i = \sqrt{\mu_i^2 + k^2}. \quad (69)$$

The transformed Hamiltonian (67) implies

$$\phi(x) = \int \frac{dx}{\sqrt{4\pi\omega_1(k)}} \left[\tilde{a}(k)e^{-i\omega_1(k)t+ikx} + \tilde{a}^\dagger(k)e^{i\omega_1(k)t-ikx}\right], \quad (70)$$

$$\chi_2(x) = \int \frac{dx}{\sqrt{4\pi\omega_2(k)}} \left[\tilde{c}(k)e^{-i\omega_2(k)t+ikx} + \tilde{c}^\dagger(k)e^{i\omega_2(k)t-ikx}\right], \quad (71)$$

where

$$\omega_1(k) = \sqrt{\mu^2 + k^2}, \quad \omega_2(k) = \sqrt{m^2 + k^2}. \tag{72}$$

The Fock Hamiltonian

$$H = \int dk \big[E_1(k) a^\dagger(k) a(k) + E_2(k) c^\dagger(k) c(k) + \tag{73}$$

$$+ \frac{g}{\sqrt{4E_1(k)E_2(k)}} \big(a^\dagger(k)c(k) + c^\dagger(k)a(k) + a^\dagger(k)c(-k) + c^\dagger(k)a(-k) \big) \big].$$

The rotation (63) and field expansions imply

$$a(k) = \cos\alpha \frac{\sqrt{E_1(k)}}{\sqrt{\omega_1(k)}} \tilde{a}(k) + \sin\alpha \frac{\sqrt{E_1 k(k)}}{\sqrt{\omega_2(k)}} \tilde{c}(k), \tag{74}$$

$$c(k) = -\sin\alpha \frac{\sqrt{E_2(k)}}{\sqrt{\omega_1(k)}} \tilde{a}(k) + \cos\alpha \frac{\sqrt{E_2 k(k)}}{\sqrt{\omega_2(k)}} \tilde{c}(k). \tag{75}$$

Inserting these to the Fock Hamiltonian (73) DOES NOT make it diagonal

$a^\dagger(k)a^\dagger(-k)$ and similar terms persist $\Rightarrow$ a Bogoliubov transformation necessary (mixes creation and annihilation operators (with opposite sign of momentum))

before explicit construction, the same derivation for the LF theory

The LF Lagrangian

$$\mathcal{L} = 2\partial_+\varphi_1\partial_-\varphi_1 - \frac{1}{2}\mu_1^2\varphi_1^2 + 2\partial_+\varphi_2\partial_-\varphi_2 - \frac{1}{2}\mu_1^2\varphi^2 - g\varphi_1\varphi_2 \qquad (76)$$

diagonalized in the same way:

$$\varphi_1 = \cos\alpha\phi + \sin\alpha\chi, \varphi_2 = -\sin\alpha\phi + \cos\alpha\chi, \qquad (77)$$

$$\alpha = \frac{1}{4i} \ln \frac{1 + i\frac{2g}{\Delta}}{1 - \frac{2g}{\Delta}}, \tag{78}$$

$$\mu^2 = \frac{1}{2}(\mu_1^2 + \mu_2)^2 - \frac{1}{2}\sqrt{g^2 + (\mu_2^2 - \mu_1^2)^2}, \tag{79}$$

$$m^2 = \frac{1}{2}(\mu_1^2 + \mu_2)^2 + \frac{1}{2}\sqrt{g^2 + (\mu_2^2 - \mu_1^2)^2}. \tag{80}$$

The corresponding LF Hamiltonian is

$$P^- = \int dx \big[\mu_1^2\varphi_1^2 + \mu_2^2\varphi_2^2 + 2\varphi_1\varphi_2\big]. \tag{81}$$

The LF field expansions are

$$\varphi_1(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}}\left[a(k^+)e^{-\frac{i}{2}k^+x^- - \frac{\mu_1^2}{k^+}x^+} + a^\dagger(k^+)e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu_1^2}{k^+}x^+}\right] \quad (82)$$

$$\varphi_2(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}}\left[c(k^+)e^{-\frac{i}{2}k^+x^- - \frac{\mu_2^2}{k^+}x^+} + c^\dagger(k^+)e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu_2^2}{k^+}x^+}\right] \quad (83)$$

with

$$[\varphi_1(x^+,x^-),\pi_1(x^+,y^-)] = [\varphi_2(x^+,x^-),\pi_2(x^+,y^-)] = \delta(x^- - y^-), \quad (84)$$

$$\pi_1 = 2\partial_-\varphi_1, \pi_2 = 2\partial_-\varphi_2. \quad (85)$$

New Lagrangian and the corresponding Hamiltionian read

$$\mathcal{L} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2 + 2\partial_+\chi\partial_-\chi - \frac{1}{2}m^2\chi^2, \tag{86}$$

$$P^- = \int dx\big[\mu^2\phi^2 + m^2\chi^2\big]. \tag{87}$$

The LF field expansions are

$$\phi(x) = \int\limits_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}}\big[\tilde{a}(k^+)e^{-\frac{i}{2}k^+x^- - \frac{\mu^2}{k^+}x^+} + \tilde{a}^\dagger(k^+)e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{\mu^2}{k}^+ x^+}\big] \tag{88}$$

$$\chi(x) = \int\limits_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}}\big[\tilde{c}(k^+)e^{-\frac{i}{2}k^+x^- - \frac{m^2}{k^+}x^+} + \tilde{c}^\dagger(k^+)e^{\frac{i}{2}k^+x^- + \frac{i}{2}\frac{m_2^2}{k^+}x^+}\big]. \tag{89}$$

After the field rotations, the original LF Fock Hamiltonian

$$P^- = \int_0^\infty \frac{dk^+}{k^+}\left[\mu_1^2 a^\dagger(k^+)a(k^+) + \mu_2^2 c^\dagger(k^+)c(k^+) + g\big(a^\dagger(k^+)c(k^+) + c^\dagger(k^+)a(k^+)\big)\right] \tag{90}$$

goes over to

$$\tilde{P}^- = \int_0^\infty \frac{dk^+}{k^+}\left[\mu^2 \tilde{a}^\dagger(k^+)\tilde{a}(k^+) + m_2^2 \tilde{c}^\dagger(k^+)\tilde{c}(k^+)\right], \tag{91}$$

$$\mu^2 = \frac{1}{2}\big(\mu_1^2 + \mu_2\big)^2 - \frac{1}{2}\sqrt{g^2 + (\mu_2^2 - \mu_1^2)^2}, \tag{92}$$

$$m^2 = \frac{1}{2}\big(\mu_1^2 + \mu_2\big)^2 + \frac{1}{2}\sqrt{g^2 + (\mu_2^2 - \mu_1^2)^2}. \tag{93}$$

as before (SL case). This is because the LF Fock operators rotate as

$$a(k^+) = \cos\alpha \tilde{a}(k^+) + \sin\alpha \tilde{c}(k^+), \tag{94}$$

$$c(k^+) = -\sin\alpha \tilde{a}(k^+) + \cos\alpha \tilde{c}(k^+), \tag{95}$$

i.e in the same way as the field operators. This occurs since there are no extra kinematical factors in the LF Fock expansions, or, in other words, the only mass dependence sits in the plane wave factors and vanishes at $x^+ = 0$

Full consistency: no mixed terms $\tilde{a}^\dagger(k)\tilde{c}^\dagger(-k)$ can be present in the LF Hamiltonian due to positivity of k^+.

Now back to the SL case - some details of the derivation and properties of the vacuum state (true physical)

Rotated Fock Hamiltonian will still mix creation and annihilation operators, complicated kinematical functions but EXACT diagonalization of the (quadratic) Hamiltonian possible, by means of a unitary operator $U(\theta)$

$$\tilde{a}(k) \to U(\theta)\tilde{a}(k)u^{-1} = \cosh\theta\tilde{a}(k) + \sinh(\theta)\tilde{c}^{\dagger}(-k),$$
$$\tilde{c}(k) \to .... \tag{96}$$

The new vacuum state schematically

$$U(\theta)HU^{-1}(\theta)|0\rangle = 0 \Rightarrow |\Omega\rangle = U^{-1}(\theta)|0\rangle. \tag{97}$$

Fock form (squeezed state)

$$|\Omega\rangle = \mathcal{N}\exp\left[\int dk\big(f(k)a^{\dagger}(k)a^{\dagger}(-k)+g(k)c^{\dagger}(k)c^{\dagger}(-k)+h(k)a^{\dagger}(k)c^{\dagger}(-k)\big]|0\rangle. \tag{98}$$

Final result: Hamiltonian diagonal in new (composite) operators – same structure as for the LF case

relation between the LF (elementary Fock) and SL (composite) field operators?

FURTHER CALCULATIONS NEEDED