

The Pion Structure from the Bethe–Salpeter Equation in Minkowski Space.

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Collaborators

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Motivation

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⇒ The approach we are pursuing is based on using the Nakanishi Integral Representation to solve the Dynamical Equations and compute observables in Minkowski space.

Summary

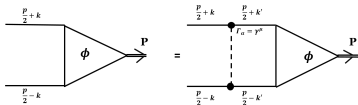
- 1 Pion as a quark-antiquark bound state
- 2 Observables
 - LF Momentum Distributions
 - EM form factor and charge radius
 - Transverse Momentum-Dependent Distributions (TMDs)
- 3 Dressed quark propagator
- 4 Conclusions and Perspectives

Pion as a quark-antiquark bound state

Bethe-Salpeter equation (0^-) :

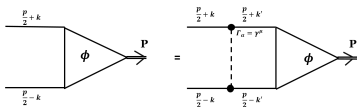
Pion as a quark-antiquark bound state

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$$\Phi(k; P) = S(k + \frac{P}{2}) \int \frac{d^4 k'}{(2\pi)^4} S^{\mu\nu}(q) \Gamma_\mu(q) \Phi(k'; P) \hat{\Gamma}_\nu(q) S(k - \frac{P}{2})$$

$$\hat{\Gamma}_\nu(q) = C \Gamma_\nu(q) C^{-1}$$

where we use: i) bare propagators for the quarks and gluons;

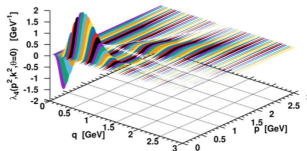
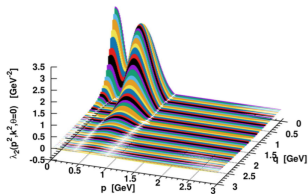
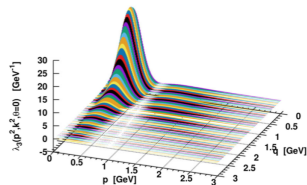
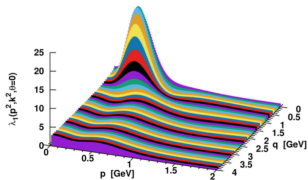
ii) ladder approximation with massive gluons,

iii) an extended quark-gluon vertex

$$S(P) = \frac{i}{\not{P} - m + i\epsilon}, \quad S^{\mu\nu}(q) = -i \frac{g^{\mu\nu}}{q^2 - \mu^2 + i\epsilon}, \quad \Gamma^\mu = ig \frac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\epsilon} \gamma^\mu,$$

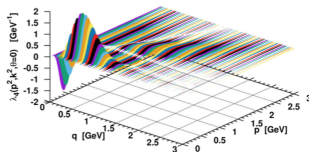
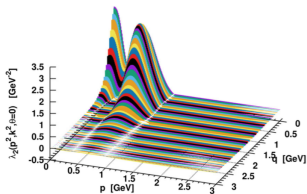
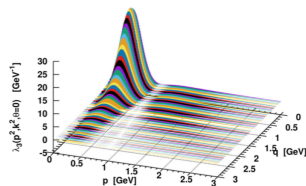
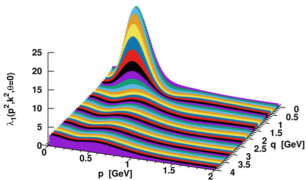
Pion as a quark-antiquark bound state

Oliveira, Frederico, WP, EPJC 80 (2020) 484



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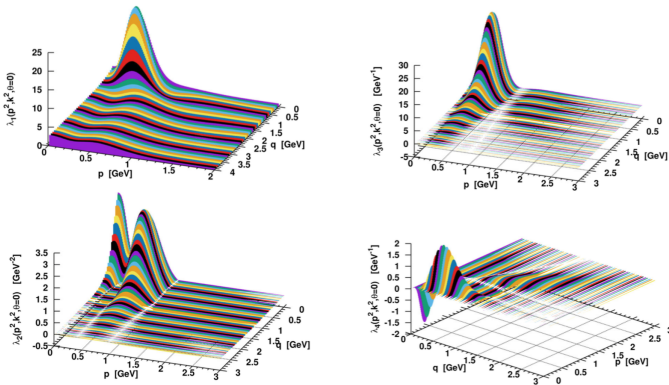
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We consider one of the Longitudinal components of the QGV

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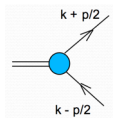


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We set the value of the scale parameter (~ 300 MeV) from the combined analysis of Lattice simulations, the Quark-Gap Equation and Slanov-Taylor identity.

Oliveira, WP, Frederico, de Melo EPJC 78(7), 553 (2018) & EPJC 79 (2019) 116 & EPJC 80 (2020) 484

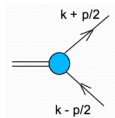
NIR for fermion-antifermion 0^- Bound State



BSA for a quark-antiquark 0^- bound state:

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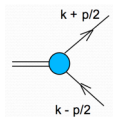


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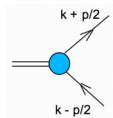
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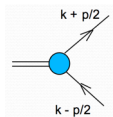
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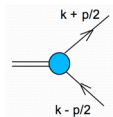
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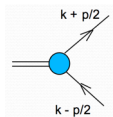
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$$\int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g_i(\gamma', z')}{[k^2 + z'p \cdot k - \gamma' - \kappa^2 + i\epsilon]^3} = \sum_j \int_{-1}^1 dz' \int_0^\infty d\gamma' \mathcal{K}_{ij}(k, p; \gamma', z') g_j(\gamma', z')$$

Projecting BSE onto the LF hyper-plane $x^+ = 0$

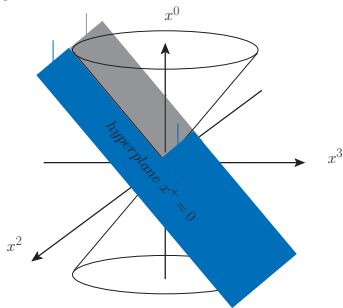
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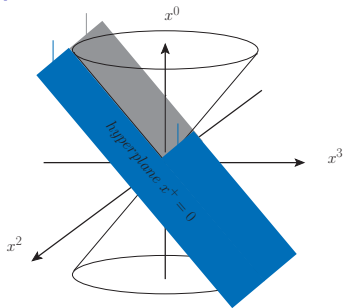
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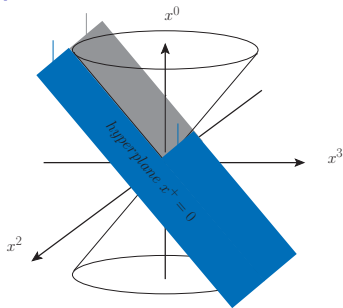


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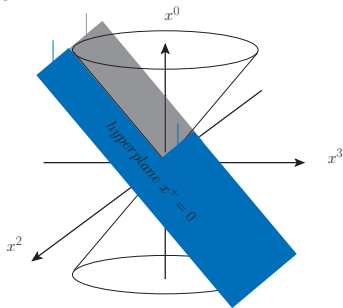
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By LF-projecting both sides of BSE (after applying the suitable traces on Dirac indexes) one gets a **coupled integral-equation system**.

The coupled integral-equation system (see also NIR+covariant LF, Carbonell and Karmanov JPA 2010) in ladder approximation, reads (cf. de Paula, et al, PRD **94**, 071901 (2016) & EPJC **77**, 764 (2017))

$$\int_0^\infty \frac{d\gamma' g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2]^2} = iMg^2 \sum_j \int_0^\infty d\gamma' \int_{-1}^1 dz' \mathcal{L}_{ij}(\gamma, z; \gamma', z') g_j(\gamma', z'; \kappa^2)$$

In ladder approximation, the Nakanishi Kernel, $\mathcal{L}_{ij}$, has an analytical expression and contains **end-point singularities** that can be treated 'a la Yan (Chang and Yan, Quantum field theories in the infinite momentum frame. II. PRD **7**, 1147 (1973)).

Singular contributions

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For two-fermion BSE

$$C_j = \int_{-\infty}^{+\infty} \frac{dk^-}{2\pi} (k^-)^j \mathcal{S}(k^-, v, z, z', \gamma, \gamma')$$

with $j = 1, 2, 3$.

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Then one can not close the arc at the infinity. We calculate the singular contribution using (Yan PRD 7 (1973) 1780):

$$\int_{-\infty}^{+\infty} dx \frac{1}{(\beta x - y \mp i\epsilon)^2} = \pm 2\pi i \frac{\delta(\beta)}{(-y \mp i\epsilon)}$$

Numerical solutions are obtained by discretizing the system using a polynomial basis, given by the Cartesian product of $\text{Laguerre}(\gamma) \times \text{Gegenbauer}(z)$. One remains with a Generalized eigenvalue problem, where a non-symmetric matrix and a symmetric one are present

$$A \vec{c} = \lambda B \vec{c}$$

N.B. the eigenvector $\vec{c}$ contains the coefficients of the expansion of the Nakanishi weight functions $g_i(z, \gamma; \kappa^2)$.

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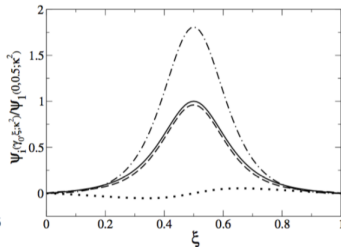
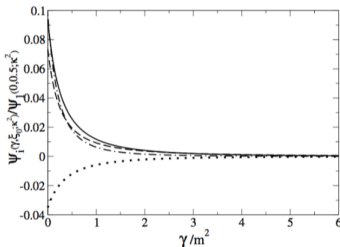
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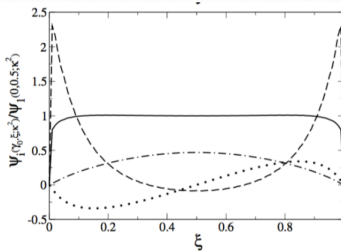
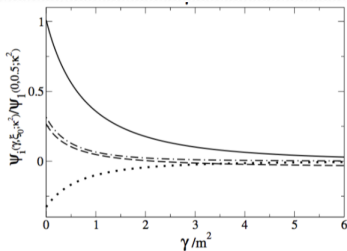
LF Amplitudes: Vector Exchange

WP, T Frederico, G Salme, M Viviani and R Pimentel, EPJC 77 (2016) 264

Weak Binding



Strong Binding



LF Momentum Distributions

LF Momentum Distributions

The fermionic field on the null-plane is given by:

$$\psi^{(+)}(\tilde{x}, x^+ = 0^+) = \int \frac{d\tilde{q}}{(2\pi)^{3/2}} \frac{\theta(q^+)}{\sqrt{2q^+}} \sum_{\sigma} \left[U^{(+)}(\tilde{q}, \sigma) b(\tilde{q}, \sigma) e^{i\tilde{q} \cdot \tilde{x}} + V^{(+)}(\tilde{q}, \sigma) d^{\dagger}(\tilde{q}, \sigma) e^{-i\tilde{q} \cdot \tilde{x}} \right],$$

where

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$$\varphi_2(\xi, \mathbf{k}_{\perp}, \sigma_i; M, J^{\pi}, J_z) = (2\pi)^3 \sqrt{N_c} 2p^+ \sqrt{\xi(1-\xi)} \langle 0 | b(\tilde{q}_2, \sigma_2) d(\tilde{q}_1, \sigma_1) | \tilde{p}, M, J^{\pi}, J_z \rangle ,$$

where $\tilde{q}_1 \equiv \{q_1^+ = M(1-\xi), -\mathbf{k}_{\perp}\}$, $\tilde{q}_2 \equiv \{q_2^+ = M\xi, \mathbf{k}_{\perp}\}$ and $\xi = 1/2 + k^+/p^+$.

LF Momentum Distributions

LF Momentum Distributions

LF valence amplitude in terms of BS amplitude is:

$$\varphi_2(\xi, \mathbf{k}_\perp, \sigma_j; M, J^\pi, J_z) = \frac{\sqrt{N_c}}{p^+} \frac{1}{4} \bar{u}_\alpha(\tilde{q}_2, \sigma_2) \int \frac{dk^-}{2\pi} \left[\gamma^+ \Phi(k, p) \gamma^+ \right]_{\alpha\beta} v_\beta(\tilde{q}_1, \sigma_1) .$$

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Anti-aligned configuration:

$$\psi_{\uparrow\downarrow}(\gamma, z) = \psi_2(\gamma, z) + \frac{z}{2} \psi_3(\gamma, \xi) + \frac{i}{M^3} \int_0^\infty d\gamma' \frac{\partial g_3(\gamma', z)/\partial z}{\gamma + \gamma' + z^2 m^2 + (1 - z^2) \kappa^2}$$

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with the LF amplitudes given by

$$\psi_i(\gamma, z) = -\frac{i}{M} \int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2) \kappa^2]^2}$$

Valence probability

Valence probability

The Fock expansion allows to restore a probabilistic framework.

The Valence Probability is:

$$P_{val} = \frac{1}{(2\pi)^3} \sum_{\sigma_1 \sigma_2} \int_{-1}^1 \frac{dz}{(1-z^2)} \int d\mathbf{k}_\perp \left| \varphi_{n=2}(\xi, \mathbf{k}_\perp, \sigma_i; M, J^\pi, J_z) \right|^2$$

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In terms of the aligned and anti-aligned LFWF, we have

$$P_{val} = \int_{-1}^1 dz \int_0^\infty \frac{d\gamma}{(4\pi)^2} \left[|\psi_{\uparrow\downarrow}(\gamma, z)|^2 + |\psi_{\uparrow\uparrow}(\gamma, z)|^2 \right],$$

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The contribution to the PDF from the LF-valence WF is

$$u_{val}(z) = \int_0^\infty \frac{d\gamma}{(4\pi)^2} \left[|\psi_{\uparrow\downarrow}(\gamma, z)|^2 + |\psi_{\uparrow\uparrow}(\gamma, z)|^2 \right]$$

Decay constant

The decay constant is defined in terms of the BSA, as follows:

$$i p^\mu f_\pi = N_c \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[\gamma^\mu \gamma^5 \Phi_\pi(k, p)] . \quad (1)$$

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which is equivalent to

$$f_\pi = i \frac{N_c}{8\pi^2} \int_0^\infty d\gamma \int_{-1}^1 dz \psi_{\uparrow\downarrow}(\gamma, z). \quad (2)$$

Quantitative results: Static properties

WP, Ydrefors, Nogueira, Frederico and Salme PRD 103 014002 (2021).

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Set	m (MeV)	B/m	μ/m	Λ/m	P_{val}	$P_{\uparrow\downarrow}$	$P_{\uparrow\uparrow}$	f_π (MeV)
I	187	1.25	0.15	2	0.64	0.55	0.09	77
II	255	1.45	1.5	1	0.65	0.55	0.10	112
III	255	1.45	2	1	0.66	0.56	0.11	117
IV	215	1.35	2	1	0.67	0.57	0.11	98
V	187	1.25	2	1	0.67	0.56	0.11	84
VI	255	1.45	2.5	1	0.68	0.56	0.11	122
VII	255	1.45	2.5	1.1	0.69	0.56	0.12	127
VIII	255	1.45	2.5	1.2	0.70	0.57	0.13	130
IX	255	1.45	1	2	0.70	0.57	0.14	134
X	215	1.35	1	2	0.71	0.57	0.14	112
XI	187	1.25	1	2	0.71	0.58	0.14	96

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The set VIII reproduces the **pion decay constant**

$$m_q = 255 \text{ MeV}, m_g = 637.5 \text{ MeV} \text{ and } \Lambda = 306 \text{ MeV}$$

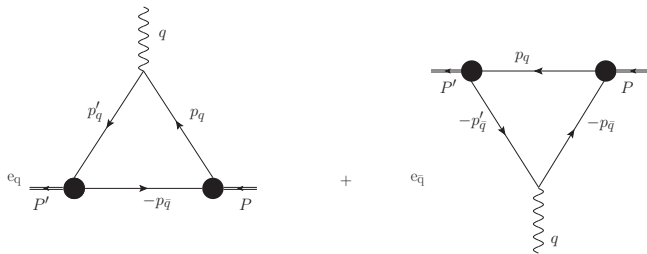
The contributions **beyond the valence** component are important, $\sim 30\%$

Pion EM form factor in ladder approximation

Ydrefors, WP, Nogueira, Frederico and Salmè PLB 820, 136494 (2021)

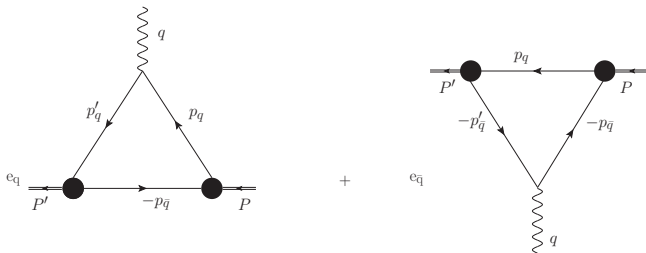
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Pion EM form factor in ladder approximation

Ydrefors, WP, Nogueira, Frederico and Salmè PLB 820, 136494 (2021)



The elastic FF is given by

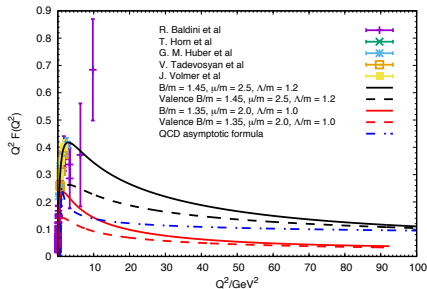
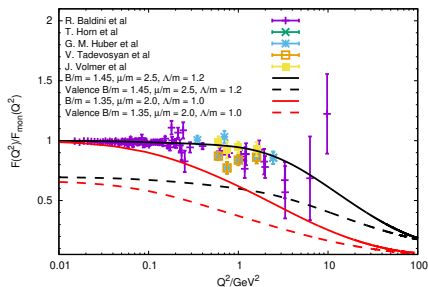
$$F(Q^2) = -i \frac{N_c}{M^2 (1 + \tau)} \int \frac{d^4 p_{\bar{q}}}{(2\pi)^4} \text{Tr} [(-\not{p}_{\bar{q}} - m) \bar{\Phi}(k'; P') (\not{P} + \not{P}') \Phi(k; P)]$$

Pion EM form factor in ladder approximation

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$$m_q = 255\text{MeV}, m_g = 637\text{MeV} \text{ and } \Lambda = 306\text{MeV}$$

Good agreement with experimental data (black solid curve).

For high Q^2 we obtain the valence dominance (dashed black curve)

Right Panel: Dash-dotted line; asymptotic expression from Brodsky-Lepage PRD 22 (1980): $Q^2 F_{\text{asy}}(Q^2) = 8\pi\alpha_s(Q^2)f_\pi^2$. Our results recover the pQCD for large Q^2

Pion charge radius

Ydrefors, WP, Nogueira, Frederico and Salmè PLB 820, 136494 (2021)

Pion charge radius

Ydrefors, WP, Nogueira, Frederico and Salmè PLB 820, 136494 (2021)

Pion charge radius and its decomposition in valence and non valence contributions.

Set	m	B/m	μ/m	Λ/m	P_{val}	f_π	r_π (fm)	r_{val} (fm)	r_{nval} (fm)
I	255	1.45	2.5	1.2	0.70	130	0.663	0.710	0.538
II	215	1.35	2	1	0.67	98	0.835	0.895	0.703

where $r_\pi^2 = -6 \left. dF_\pi(Q^2)/dQ^2 \right|_{Q^2=0}$

$P_{val(nval)} r_{val(nval)}^2 = -6 \left. dF_{val(nval)}(Q^2)/dQ^2 \right|_{Q^2=0}$

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The set I is in fair agreement with the PDG value:

$$r_\pi^{PDG} = 0.659 \pm 0.004 \text{ fm}$$

Pion Transverse Momentum-Dependent Distributions

One can define the T-even subleading quark uTMDs, starting from the decomposition of the pion correlator (Mulders and Tangerman, Nucl. Phys. B 461, 197 (1996)).

twist -2 uTMD:

$$f_1^q(\gamma, \xi) = \frac{N_c}{4} \int d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3} e^{i[\tilde{\mathbf{k}} \cdot \tilde{\mathbf{y}}]} \langle P | \bar{\psi}_q(-\frac{y}{2}) \gamma^+ \psi_q(\frac{y}{2}) | P \rangle \big|_{y^+=0}$$

twist-3 uTMD

$$\frac{M}{P^+} e^q(\gamma, \xi) = \frac{N_c}{4} \int d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3} e^{i[\tilde{\mathbf{k}} \cdot \tilde{\mathbf{y}}]} \langle P | \bar{\psi}_q(-\frac{y}{2}) \hat{1} \psi_q(\frac{y}{2}) | P \rangle \big|_{y^+=0}$$

and

$$\frac{M}{P^+} f^{\perp q}(\gamma, \xi) = \frac{N_c M}{4|\mathbf{k}_\perp|^2} \int d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3} e^{i[\tilde{\mathbf{k}} \cdot \tilde{\mathbf{y}}]} \langle P | \bar{\psi}_q(-\frac{y}{2}) \mathbf{k}_\perp \cdot \boldsymbol{\gamma}_\perp \psi_q(\frac{y}{2}) | P \rangle \big|_{y^+=0}$$

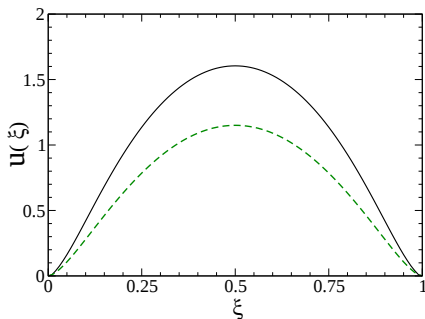
with $\tilde{\mathbf{k}} \cdot \tilde{\mathbf{y}} = \xi P^+ y^- / 2 - \mathbf{k}_\perp \cdot \mathbf{y}_\perp$.

Parton distribution function

W. de Paula et al., PRD **105**, L071505 (2022).

From the charge-symmetric expression for the leading-twist TMD $f_1^S(\gamma, \xi)$, one gets the PDF at the initial scale $u(\xi)$

$$f_1^{S(AS)}(\gamma, \xi) = \frac{f_1^q(\gamma, \xi) \pm f_1^{\bar{q}}(\gamma, 1 - \xi)}{2} \Rightarrow u(\xi) = \int_0^\infty d\gamma f_1^S(\gamma, \xi).$$



Solid line: full calculation of the BSE at the model scale

Dashed line: The LF valence contribution .

At the initial scale, for $\xi \rightarrow 1$, the exponent of $(1 - \xi)^{\eta_0}$ is $\eta_0 = 1.4$.

Parton distribution function II

WP, Frederico and Salmè - EPJC 83(2023) 985

Low order Mellin moments at scales $Q = 2.0$ GeV and $Q = 5.2$ GeV.

	BSE ₂	LQCD ₂	BSE ₅	LQCD ₅
$\langle x \rangle$	0.259	0.261 ± 0.007	0.221	0.229 ± 0.008
$\langle x^2 \rangle$	0.105	0.110 ± 0.014	0.082	0.087 ± 0.009
$\langle x^3 \rangle$	0.052	0.024 ± 0.018	0.039	0.042 ± 0.010
$\langle x^4 \rangle$	0.029		0.021	0.023 ± 0.009
$\langle x^5 \rangle$	0.018		0.012	0.014 ± 0.007
$\langle x^6 \rangle$	0.012		0.008	0.009 ± 0.005

LQCD, $Q = 2.0$ GeV: $\langle x \rangle$ - Alexandrou et al PRD 103, 014508 (2021)
 $\langle x^2 \rangle$ and $\langle x^3 \rangle$ - Alexandrou et al PRD 104, 054504 (2021)

LQCD, $Q = 5.0$ GeV: $\langle x \rangle$ - Alexandrou et al PRD 103, 014508 (2021)

N.B. following Cui et al EPJC 2020 80 1064, lowest order DGLAP equations used for evolution. One needs:

Hadronic scale and effective charge for dealing with DGLAP

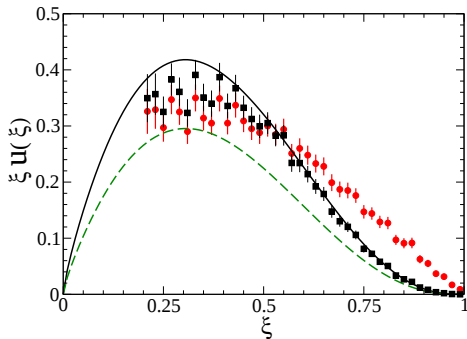
$Q_0 = 0.330 \pm 0.030$ GeV

Within the error, we choose $Q_0 = 0.360$ GeV to fit the first Mellin moment.

Parton distribution function III

WP, Frederico and Salmè - EPJC 83(2023) 985

Comparison with the data at 5.2 GeV scale



Solid line: full calculation of the BSE evolved from the initial scale $Q_0 = 0.360$ GeV to $Q = 5.2$ GeV

Dashed line: The evolved LF valence contribution

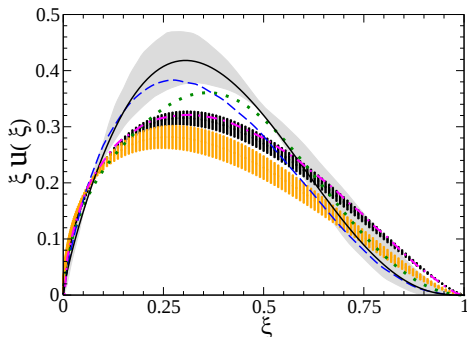
Full dots: experimental data from E615

Full squares: reanalyzed experimental data from Aicher et al PRL 105, 252003 (2010) evolved to $Q = 5.2$ GeV

Parton distribution function IV

WP, Frederico and Salmè - EPJC 83(2023) 985

Comparison with other theoretical calculations



Solid line: full calculation of the BSE evolved from the initial scale $Q_0 = 0.360$ GeV to $Q = 5.2$ GeV

Dashed line: DSE calculation from Cui et al, Eur. Phys. J. A 58, 10 (2022)

Dash-dotted line: DSE calculation with dressed quark-photon vertex from Bednar et al PRL 124, 042002 (2020)

Dotted line: BLFQ collaboration, PLB 825, 136890 (2022)

Gray area: LQCD results from C. Alexandrou et al (2021)

Black and Orange vertical lines from JAM collaboration, private communication.

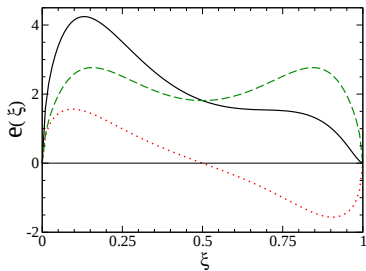
For the evolved $\xi u(\xi)$, the exponent of $(1 - \xi)^{\eta_5}$ is $\eta_5 = 2.94$, when $\xi \rightarrow 1$,

LQCD: Alexandrou et al PRD 104, 054504 (2021) obtained 2.20 ± 0.64

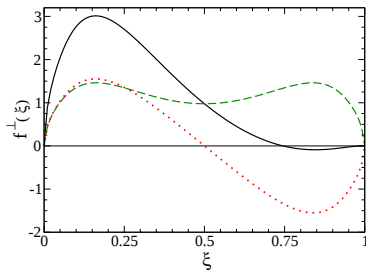
Cui et al EPJA 58, 10 (2022) obtained 2.81 ± 0.08

Transverse Momentum-Dependent Distributions II

WP, Frederico and Salmè - EPJC 83(2023) 985



Solid line: quark twist-3 uTMD $e(\xi)$
 Dashed line: Sym. twist-3 uTMD $e^S(\xi)$
 Dotted: AS twist-3 uTMD $e^{AS}(\xi)$

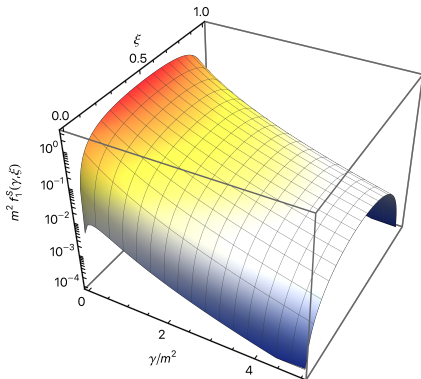


Solid line: quark twist-3 uTMD $f^\perp(\xi)$
 Dashed line: Sym. twist-3 uTMD $f^{\perp S}(\xi)$
 Dotted: AS twist-3 uTMD $f^{\perp AS}(\xi)$

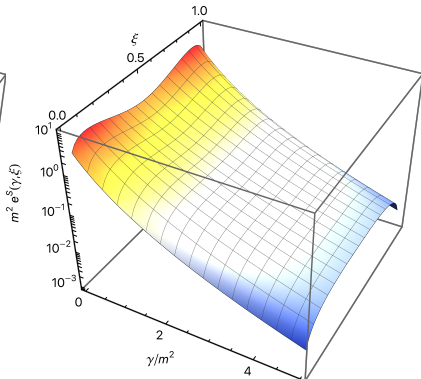
The corresponding symmetric and antisymmetric collinear PDFs are:

$$e^{S(AS)}(\xi) = \int_0^\infty d\gamma e^{S(AS)}(\gamma, \xi), \quad f^{\perp S(AS)}(\xi) = \int_0^\infty d\gamma f^{\perp S(AS)}(\gamma, \xi)$$

For the quark ones: $e^q(\xi) = e^S(\xi) + e^{AS}(\xi)$ and $f^{\perp q}(\xi) = f^{\perp S}(\xi) + f^{\perp AS}(\xi)$



Twist-2 uTMD $f_1^S(\gamma, \xi)$



Twist-3 uTMD $e^S(\gamma, \xi)$

Twist-2

- i) the peak at $\xi = 0.5$ for any γ/m^2
- ii) the vanishing values at the end-points
- iii) the order of magnitude fall-off already for $\gamma/m^2 > 2$

Similar behavior in comparison with DSE calculations (Shi, Bednar, Cloët, PRD 101(7), 074014 (2020))

Different behavior in comparison to "LF constituent model" (Pasquini, Schweitzer, PRD 90(1), 014050 (2014)) and "LF holographic models" (Bacchetta, Cotogno, Pasquini, PLB 771, 546 (2017)).

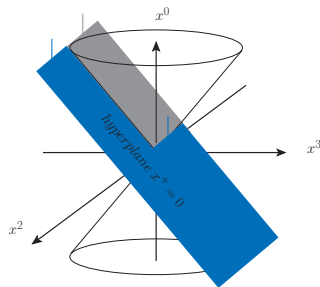
Twist-3

Double-hump: smooth for larger γ/m^2 .

A view of the pion valence wave function on the null-plane

W. de Paula, et al, PRD 103, 014002 (2021) The probability distribution of the quarks inside the pion, sitting on the the hyperplane $x^+ = 0$, tangent to the light-cone, is evaluated in the space given by the Cartesian product of the *loffe-time* and the plane spanned by the transverse coordinates $\mathbf{b}_\perp$.

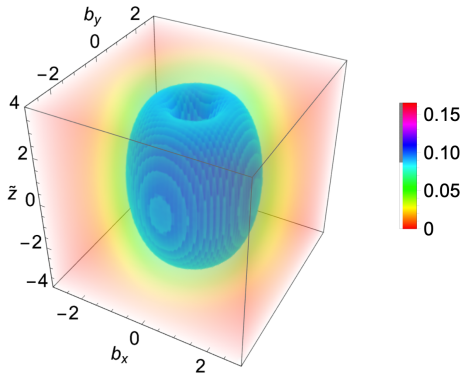
Why? Adopting the configuration space, a complementary description of the space-time structure of the hadron can be determined. The *loffe-time* is useful for studying the relative importance of short and long light-like distances.



The covariant definition of the loffe-time is $\tilde{z} = x \cdot P_{target}$, and it becomes $\tilde{z} = x^- P_{target}^+ / 2$ on the hyperplane $x^+ = 0$

The pion on the light-cone

Density plot of $|\mathbf{b}_\perp|^2 |\psi(\tilde{z}, b_x, b_y)|^2$, with $\psi(\tilde{z}, b_x, b_y)$ obtained from our solutions of the ladder Bethe-Salpeter equation [W. de Paula et al PRD 103, (2021) 014002]

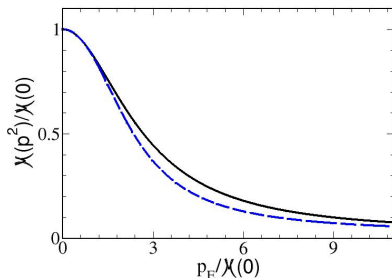


$\tilde{z} \equiv$ loffe-time

$\{b_x, b_y\} \equiv$ transverse coordinates

BS with Dressed quark propagator

After completing the investigation of the pion BSE with fixed-mass quark, i.e. a $q\bar{q}$ bound system, we are addressing the running-mass case



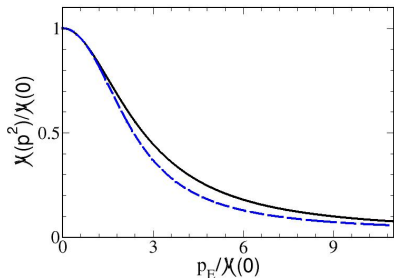
The quark running-mass, $\mathcal{M}(p^2)$, as a function of the Euclidean momentum $p_E = \sqrt{-p^2}$, in units of the IR mass $\mathcal{M}(0) = 0.344$ GeV. Solid line: our model. Dashed line: accurate fit of the LQCD calculations.

We consider the following ansatz for the quark running-mass:

$$\mathcal{M}(p_E^2) = m_0 - m^3/(p_E^2 - \lambda^2)$$

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We consider the following ansatz for the quark running-mass:

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From Lattice simulations: bare quark mass $m_0 = 8$ MeV and $\mathcal{M}(0) = 0.344$ GeV.

Our fit (solid line): $m = 0.648$ GeV and $\lambda = 0.9$ GeV.

Lattice data (dashed line) from: Oliveira, Silva, Skullerud and Sternbec, PRD 99 (2019) 094506

0^- Bound State with Running quark mass function

A Castro, WP, Ydrefors, Frederico, Salmè - PLB 845(2023) 138159

0⁻ Bound State with Running quark mass function

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Dressed quark propagator: $S(p) = S^V(p^2)\not{p} + S^S(p^2)$

Integral Representation: $S^V(p^2) = \int_0^\infty ds \frac{\rho^V(s)}{p^2 - s + i\epsilon}$; $S^S(p^2) = \int_0^\infty ds \frac{\rho^S(s)}{p^2 - s + i\epsilon}$

0^- Bound State with Running quark mass function

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Using the Nakanishi integral representation for $\phi_i(k, p)$, performing the loop integral and projecting onto the LF, one obtains the BSE as

$$\int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{\left[\gamma + z^2 M^2/4 + \gamma' + \kappa^2 - i\epsilon \right]^2} = \frac{\alpha}{2\pi} \\ \times \sum_j \int_{-1}^1 dz' \int_0^\infty d\gamma' \mathcal{L}_{ij}(\gamma, z; \gamma', z') g_j(\gamma', z').$$

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Phenomenological model: $\mathcal{M}(p^2) = m_0 - \frac{m^3}{p^2 - \lambda^2 + i\epsilon}$

$$\rho^{S(V)}(s) = \sum_{a=1}^3 R_a^{S(V)} \delta(s - m_a^2),$$

where $R_a^{S(V)}$ are the **residues**, that read

$$R_a^V = \frac{(\lambda^2 - m_a^2)^2}{(m_a^2 - m_b^2)(m_a^2 - m_c^2)},$$

$$R_a^S = R_a^V \mathcal{M}(m_a^2),$$

with the indices $\{a, b, c\}$ following the cyclic permutation $\{1, 2, 3\}$.

$\mathcal{M}(0)$	i	m_i	R_i^V	R_i^S
[GeV]		[GeV]		[GeV]
	1	0.4696	3.7784	1.7743
0.344	2	0.5733	-2.8863	-1.6546
	3	1.0349	0.1079	-0.1116

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0^- Bound State with Running quark mass function

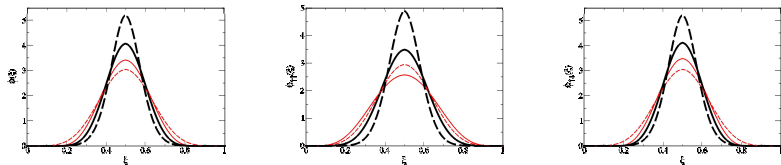
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Longitudinal momentum distribution

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Longitudinal momentum distribution



Parameters: $\Lambda = 0.12$ GeV, $\mu = 0.469$ GeV.

Thick solid line: running mass model for $M = 0.653$ GeV.

Thick dashed Line: fixed quark mass (344 MeV) for $M = 0.653$ GeV.

Thin solid line: running mass model for $M = 0.516$ GeV.

Thin dashed line: fixed quark mass (344 MeV) for $M = 0.516$ GeV.

0^- Bound State with Running quark mass function

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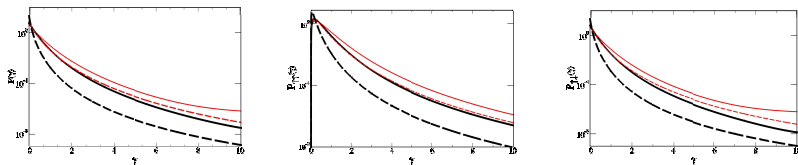
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Transverse momentum distribution

0^- Bound State with Running quark mass function

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Transverse momentum distribution



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Dressing the Quark: Schwinger-Dyson equation

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The diagram illustrates the Schwinger-Dyson equation for a quark vertex. On the left, a horizontal line with a black dot (vertex) is labeled with momentum p above it and -1 above the dot. This is set equal to the sum of two terms. The first term is a horizontal line with a double arrow (representing a ghost) and is labeled with momentum p above it and -1 above the line. The second term is a loop diagram consisting of a horizontal line with two black dots (vertices) labeled with momentum k below the right dot. A wavy line (representing a massive vector boson) connects the two dots in a loop, with the momentum $p-k$ labeled above the right half of the loop.

Dressing the Quark: Schwinger-Dyson equation

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The model: Bare vertices, massive vector boson, Pauli-Villars regulator

$$p \text{---} \bullet \text{---}^{-1} = p \text{---} \Rightarrow \text{---}^{-1} + \text{loop}(p, k)$$

The rainbow ladder Schwinger-Dyson equation in **Minkowski space** is:

$$S_q^{-1}(k) = \not{k} - m_B + ig^2 \int \frac{d^4 q}{(2\pi)^4} \Gamma_\mu(q, k) S_q(k - q) \gamma_\nu D^{\mu\nu}(q),$$

where m_B is the **quark bare mass** and g is the coupling constant.

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$$\text{Dressed propagator} = \text{Bare propagator} + \text{Loop diagram}$$

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The massive gauge boson is given by

$$D^{\mu\nu}(q) = \frac{1}{q^2 - m_g^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1 - \xi)q^\mu q^\nu}{q^2 - \xi m_g^2 + i\epsilon} \right],$$

where we have introduced an effective gluon mass m_g , as suggested by LQCD calculations (see *Dudal, Oliveira and Silva, PRD 89 (2014) 014010*).

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The dressed fermion propagator is

$$S_q(k) = \left[\not{k} A(k^2) - B(k^2) + i\epsilon \right]^{-1}.$$

Schwinger-Dyson equation in Rainbow ladder truncation

The vector and scalar **self-energies** are given by the **KLR**, respectively as:

$$\begin{aligned} A(k^2) &= 1 + \int_0^\infty ds \frac{\rho_A(s)}{k^2 - s + i\epsilon}, \\ B(k^2) &= m_B + \int_0^\infty ds \frac{\rho_B(s)}{k^2 - s + i\epsilon}. \end{aligned}$$

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The quark propagator can also be written as:

$$S_q(k) = R \frac{\not{k} + \overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \not{k} \int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon},$$

where $\overline{m}_0$ is the **renormalized mass**.

Schwinger-Dyson equation in Rainbow ladder truncation

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where $\overline{m}_0$ is the **renormalized mass**.

$$\not{k}A(k^2) - B(k^2) = ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q)\gamma_\nu}{q^2 - m_g^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1 - \xi) q^\mu q^\nu}{q^2 - \xi m_g^2 + i\epsilon} \right]$$

Gauge fixing

$$- i g^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q)\gamma_\nu}{q^2 - \Lambda^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1 - \xi) q^\mu q^\nu}{q^2 - \xi \Lambda^2 + i\epsilon} \right]$$

Pauli-Villars regulator

Schwinger-Dyson equation in Rainbow ladder truncation

- Parameters: $\alpha = \frac{g^2}{4\pi}$, Λ , m_g , $\overline{m}_0$.

- Spectral densities are obtained from the IR of the self-energy:

$$\rho_A(\gamma) = -\frac{1}{\pi} \text{Im} [A(\gamma)]$$

$$\rho_B(\gamma) = -\frac{1}{\pi} \text{Im} [B(\gamma)]$$

- Solutions of DSE obtained writing the trivial relation $S_f^{-1} S_f = 1$ in a suitable form:

$$\frac{R}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{\gamma - s + i\epsilon} = \frac{A(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$
$$\frac{R \overline{m}_0}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{\gamma - s + i\epsilon} = \frac{B(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$

Schwinger-Dyson equation in Rainbow ladder truncation

Fermion DSE solution

$$\begin{aligned}\rho_A(\gamma) &= R\mathcal{K}_{0A}^\xi(\gamma, \bar{m}_0^2, m_g^2) \\ &+ \int_0^\infty ds \mathcal{K}_A^\xi(\gamma, s, m_g^2) \rho_v(s) - [m_g \rightarrow \Lambda] \\ \rho_B(\gamma) &= R\bar{m}_0 \mathcal{K}_{0B}^\xi(\gamma, \bar{m}_0^2, m_g^2) \\ &+ \int_0^\infty ds \mathcal{K}_B^\xi(\gamma, s, m_g^2) \rho_s(s) - [m_g \rightarrow \Lambda]\end{aligned}$$

- **Driving term:**

$$\mathcal{K}_{0A(0B)}^\xi = K_{A(B)} + m_g^{-2} \bar{K}_{A(B)}^\xi$$

- **Kernel:**

$$\begin{aligned}\mathcal{K}_A^\xi(\gamma, s, m_g^2) &= K_A(\gamma, s, m_g^2) \Theta(s - (\bar{m}_0 + m_g)^2) \\ &+ m_g^{-2} \bar{K}_A^\xi(\gamma, s, m_g^2) \Theta(s - (\bar{m}_0 + \sqrt{\xi} m_g)^2)\end{aligned}$$

Connection Formulas

$$\begin{aligned}f_A(\gamma) &= 1 + \int_0^\infty ds \frac{\rho_A(s)}{\gamma - s} \\ f_B(\gamma) &= m_B + \int_0^\infty ds \frac{\rho_B(s)}{\gamma - s} \\ d(\gamma) &= \left[\gamma f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma) \right]^2 \\ &+ 4\pi^2 \left[\gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma) \right]^2\end{aligned}$$

$$\begin{aligned}\rho_v(\gamma) &= -2 \frac{f_A(\gamma)}{d(\gamma)} [\gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma)] \\ &+ \frac{\rho_A(\gamma)}{d(\gamma)} [\gamma f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma)] \\ \rho_s(\gamma) &= -2 \frac{f_B(\gamma)}{d(\gamma)} [\gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma)] \\ &+ \frac{\rho_B(\gamma)}{d(\gamma)} [\gamma f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma)]\end{aligned}$$

Schwinger-Dyson equation in Rainbow ladder truncation

Feynman gauge kernel ($\xi = 1$):

$$K_A(\gamma, s, m_g^2) = -\frac{\alpha}{4\pi} \frac{\gamma - m_g^2 + s}{\gamma^2} \sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s} \Theta[\gamma - (m_g + \sqrt{s})^2]$$

$$K_B(\gamma, s, m_g^2) = -\frac{\alpha}{4\pi} \frac{4}{\gamma} \sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s} \Theta[\gamma - (m_g + \sqrt{s})^2]$$

Remaining arbitrary ξ -gauge contribution:

$$\bar{K}_A^\xi(\gamma, s, m_g^2) = -\frac{\alpha}{4\pi} \frac{(\gamma - s)^2 - m_g^2(\gamma + s)}{2\gamma^2} \sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}$$

$$\times \Theta[\gamma - (m_g + \sqrt{s})^2] - [m_g^2 \rightarrow \xi m_g^2]$$

$$\bar{K}_B^\xi(\gamma, s, m_g^2) = \frac{\alpha m_g^2}{4\pi} \frac{\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}}{\gamma} \Theta[\gamma - (m_g + \sqrt{s})^2] - [m_g^2 \rightarrow \xi m_g^2]$$

Bare mass:

$$m_B = \bar{m}_0 + \bar{m}_0 \int_0^\infty ds \frac{\rho_A(s)}{\bar{m}_0^2 - s} - \int_0^\infty ds \frac{\rho_B(s)}{\bar{m}_0^2 - s}$$

Large Coupling Regime: Phenomenological Model

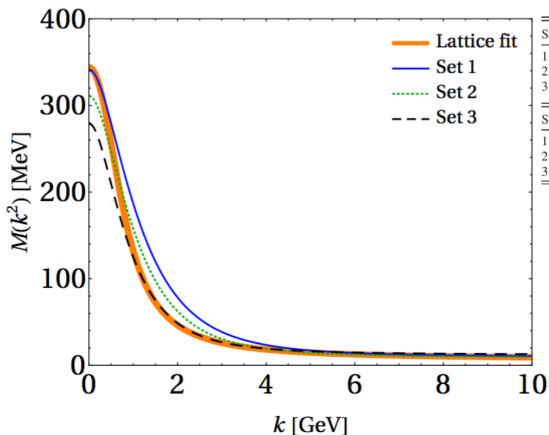
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We can calibrate the model to reproduce Lattice Data for $M(p^2)$

Large Coupling Regime: Phenomenological Model

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Set	$\bar{m}_0$ (GeV)	m_g (GeV)	Λ (GeV)	α
1	0.42	0.84	1.20	19.70
2	0.38	0.78	1.10	20.30
3	0.35	0.60	1.00	13.25

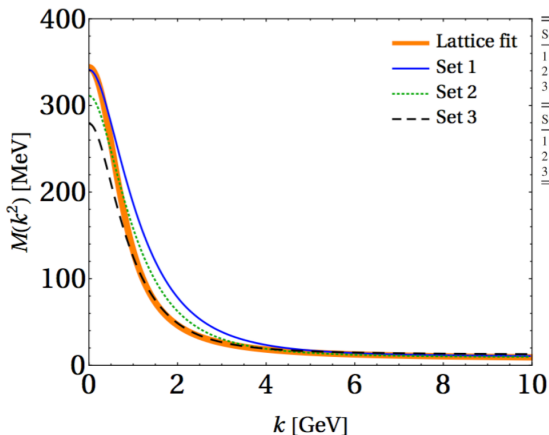
Set	(Outputs)	m_B (MeV)	R
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Lattice data from: Oliveira, Silva, Skullerud and Sternbec, PRD 99 (2019) 094506

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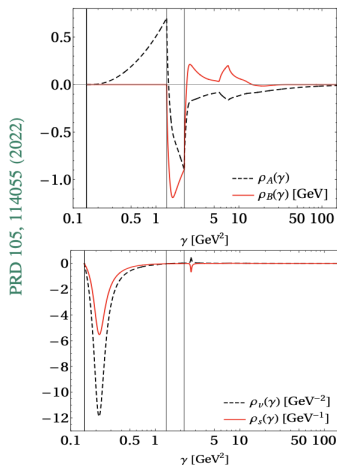
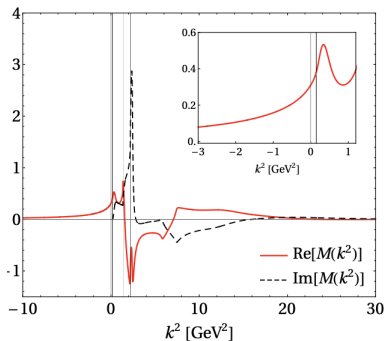
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Next step: To use a solution of the DSE to obtain the **Fermion-Antifermion bound state**

Large Coupling Regime: Phenomenological Model

Duarte, Frederico, WP, Ydrefors PRD 105, 114055 (2022)



Conclusions and Perspectives

- The near future will offer an innovative view of the dynamics inside the hadrons, thanks to the experimental activity planned at the Electron-ion colliders, and plenty of measurements pointing to the 3D tomography of hadrons will become available.
- For the pion, many results, EM form factor, PDF, TMDs, Ioffe-time $\times$ transverse plane distribution, have been obtained by using the ladder-approximation of the $q\bar{q}$ -BSE.
- The 3D imaging is in line with the goal of the future Electron Ion Collider.
- The pion has an important role, given its dual nature: $q\bar{q}$ bound-system and Goldstone boson. Our aim is to implement a framework analogous to the one already developed in Euclidean space.
- Minkowski space, phenomenological investigations, once the approach composed by BSE and gap-equations will be fully available, could offer fresh insights in hadron dynamics and possibly implement an interplay with well-established lattice and continuous QCD communities.