

# Minkowski space description of the nucleon and pion

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## Thanks to my Collaborators:

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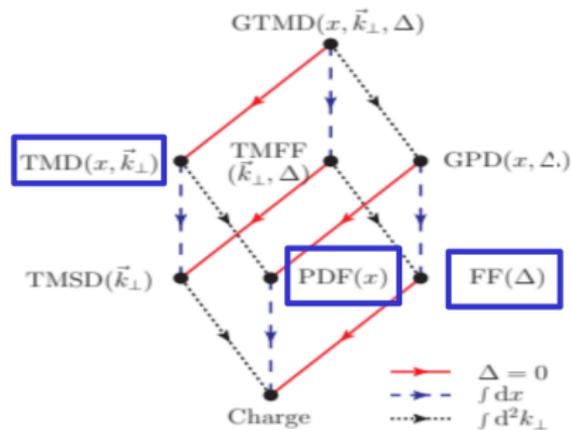
“Relativistic Few-body Systems in Minkowski space”

Nucleon

Pion

# How to get the details?

## Observables associated with the hadron structure



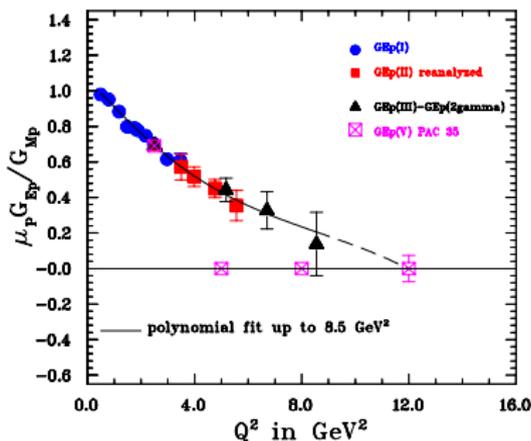
Lorcé, Pasquini, Vanderhaeghen JHEP05(2011)041

- SL form factor, PDF, TMD & 3D image & TL Obs's

# Motivations for intense theoretical efforts

Forthcoming 12 GeV Experiment at TJLAB

[hallweb.jlab.org/collab/PAC/PAC32/PR12-07-109-Ratio.pdf](http://hallweb.jlab.org/collab/PAC/PAC32/PR12-07-109-Ratio.pdf)



Sachs Form Factors

$$G_E^N(Q^2) = F_1^N(Q^2) - \frac{Q^2}{4M_N^2} \kappa_N F_2^N(Q^2) \quad G_M^N(Q^2) = F_1^N(Q^2) + \kappa_N F_2^N(Q^2)$$

$$G_M^P(Q^2) > 0 \quad \text{for} \quad 0 > Q^2 > 30 \text{ (GeV/c)}^2$$

# Proton time-like electromagnetic form factors

- Periodic Interference Structures in the TL FF: Bianconi & Tomasi-Gustafsson, PRL 114, 232301 (2015)
- New fit of TL FF: Tomasi-Gustafsson, Bianconi & Pacetti, PRC 103,035203(2021)

$$F_p(s)^2 = \frac{2\tau |G_M(s)|^2 + |G_E(s)|^2}{2\tau + 1}$$

NEW FIT OF TIMELIKE PROTON ELECTROMAGNETIC ...

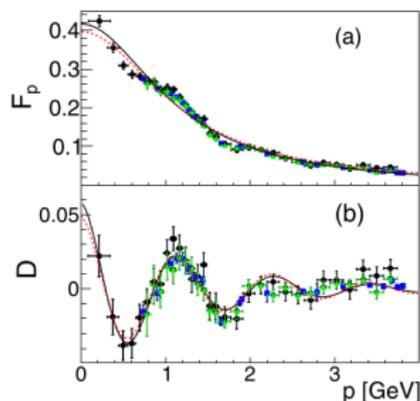
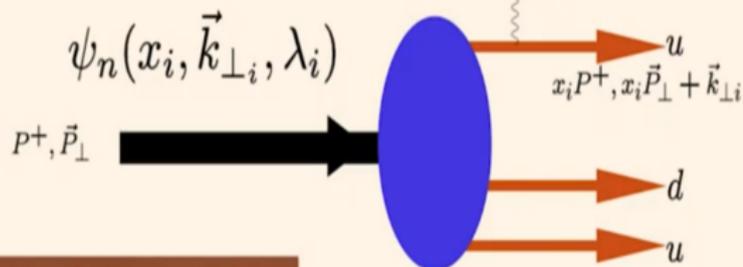


FIG. 1. (a) TL proton generalized FF as a function of  $p$  from the data of BaBar, Ref. [8] (black circles), BESIII-ISR [9] (blue squares), and BESIII-SC [11] (green triangles) with the regular background fit with Eq. (7) (black solid line); (b) data after subtraction, fitted with Eq. (8) (black solid line). For comparison the fit from Ref. [17] (red dashed lines) is also shown.

$$x = \frac{k^+}{P^+} = \frac{k^0 + k^3}{P^0 + P^3}$$



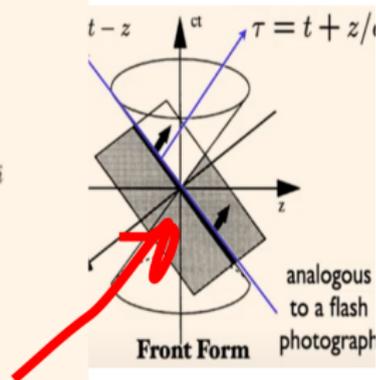
## Dirac's Front Form

*Measurements of hadron LF wavefunction are at fixed LF time*

*Like a flash photograph*

$$x_{bj} = x = \frac{k^+}{P^+}$$

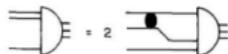
Credits Stanley Brodsky



$$|proton\rangle = |3q\rangle + |4q\ qb\rangle + |3q\ g\rangle + |3q\ 2g\rangle + \dots$$

# Nucleon Structure in Minkowski space

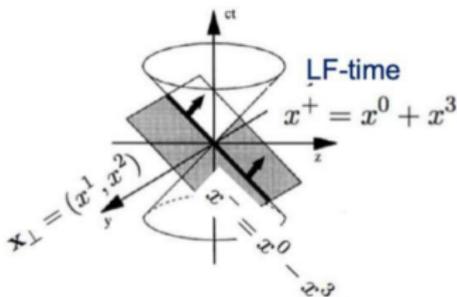
& LF Fock space decomposition of the nucleon state



$$v(q, p) = 2iF(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m^2 + i\epsilon]}$$

$$\times \frac{i}{[(p - q - k)^2 - m^2 + i\epsilon]} v(k, p).$$

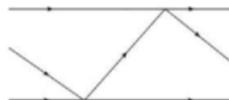
T. Frederico, Phys. Lett. B 282 (1992) 409.



$$|proton\rangle = |3q\rangle + |4q qb\rangle + |5q 2qb\rangle + \dots$$

E. Yudofors et al. / Physics Letters B 770 (2017) 131–137

**|3q>**



**|4q qb>**



Fig. 2. The three-body LF graphs obtained by time-ordering of the Feynman graph shown in right panel of Fig. 1.

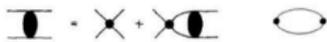
**|4q qb>**



....

Fig. 3. Examples of many-body intermediate state contributions to the LF three-body forces.

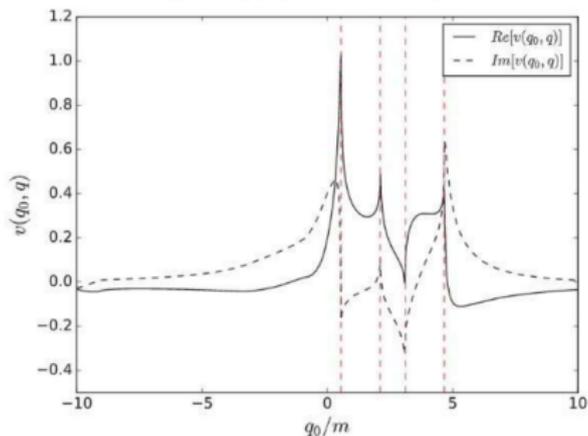
# Three-body vertex function in Minkowski space



$$\text{Diagram} = 2 \times \text{Diagram}$$

$$v(q, p) = 2iF(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m^2 + i\epsilon]} \times \frac{i}{[(p - q - k)^2 - m^2 + i\epsilon]} v(k, p).$$

E. Ydrefors et al. / Physics Letters B 791 (2019) 276–280



**Fig. 1.** The vertex function,  $v(q_0, q_T = 0.5m)$  with respect to  $q_0$  for the input parameters  $am = -1.5$  and  $B_3/m = 0.395$ . The analytical positions of the peaks, given in Eq. (13), are shown with dashed-red lines.

**Too challenging numerically!**

Further details: Ydrefors, Alvarenga Nogueira, Karmanov, TF, PRD 101 (2020) 096018

## LF dynamical model: valence proton wave function

- We consider a Light-front dynamical three-body model for the proton valence wave function (ultimately including the full BS amplitude):

$$\longrightarrow \Psi_3(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, x_3, \vec{k}_{3\perp}) = \frac{\Gamma(x_1, \vec{k}_{1\perp}) + \Gamma(x_2, \vec{k}_{2\perp}) + \Gamma(x_3, \vec{k}_{3\perp})}{\sqrt{x_1 x_2 x_3 (M_N^2 - M_0^2(x_1, \vec{k}_{1\perp}, x_2, \vec{k}_{2\perp}, x_3, \vec{k}_{3\perp}))'}}$$

$$M_0^2(x_1, \vec{k}_{1\perp} \dots) = \sum_{i=1}^3 (k_{i\perp}^2 + m^2) / x_i$$

- Fock basis truncated to valence order and spin degree-of-freedom not included.
- Valence three-body regularized LF equation:

$$\longrightarrow \Gamma(x, k_\perp) = \frac{\mathcal{F}(M_{12}^2)}{(2\pi)^3} \int_0^{1-x} \frac{dx'}{x'(1-x-x')} \int_0^\infty d^2 k'_\perp \left[ \frac{1}{M_0^2 - M_N^2} - \frac{1}{M_0^2 + \mu^2} \right] \Gamma(x', k'_\perp)$$

where  $\mu$  is a cut-off,  $k_\perp$  transverse momentum,  $x$  momentum fraction of spectator and

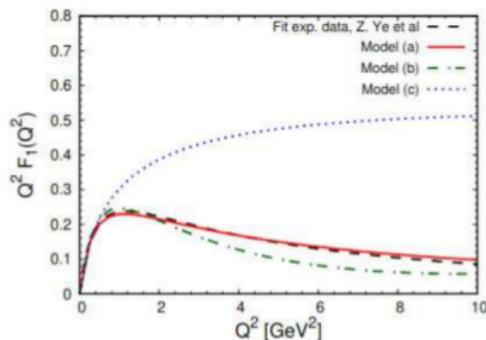
$$M_0^2 = (k_\perp^2 + m^2) / x' + (k_\perp^2 + m^2) / x + ((\vec{k}'_\perp + \vec{k}_\perp)^2 + m^2) / (1 - x - x')$$

- The cut-off  $\mu$  avoids the unphysical solution  $M_3^2 < 0$ , and enhances the IR with respect to UV.
- The quark-quark transition amplitude has a pole representing the s-wave diquark introduced through a zero-range effective interaction between two constituent quarks.

E. Ydrefors and TF, PRD 104 (2021) 114012; PLB 838 (2023) 137732

TF and G.Salme, Few Body Syst. 49 (2011) 163 “Projecting the Bethe-Salpeter Equation onto the Light-Front and back: A Short Review”

## Electromagnetic form factor



Model	$m$ [MeV]	$a$ [ $m^{-1}$ ]	$\mu/m$	$M_{dq}$ [MeV]
(a)	366	2.70	1	644
(b)	362	3.60	$\infty$	682
(c)	317	-1.84	$\infty$	-

## Valence Proton PDF @ initial and experimental scale

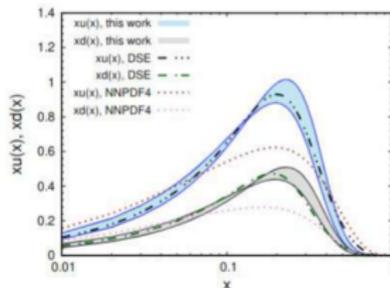
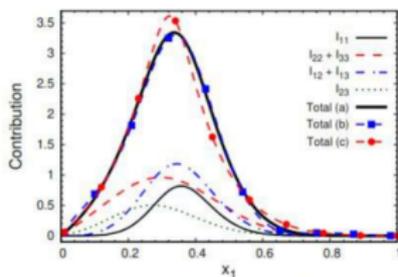
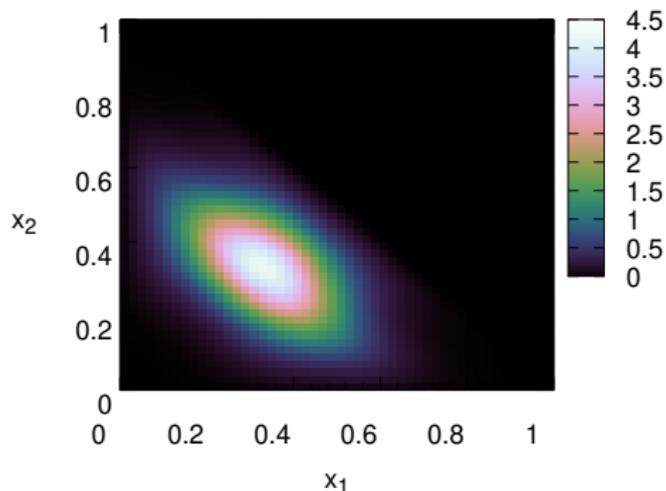


Figure: [Left] proton PDF at initial scale  $\int_0^1 dx_1 f(x_1) = 1$ . [Right] Valence  $u$ -quark and  $d$ -quark PDFs evolved to  $Q = 3.097$  GeV, compared with the DSE results of Lu et al (2203.00753 [hep-ph]) and the results of the NNPDF4 global fit. The shaded areas indicate the uncertainty with respect to the initial scale  $Q_0 = 0.33 \pm 0.03$  GeV.

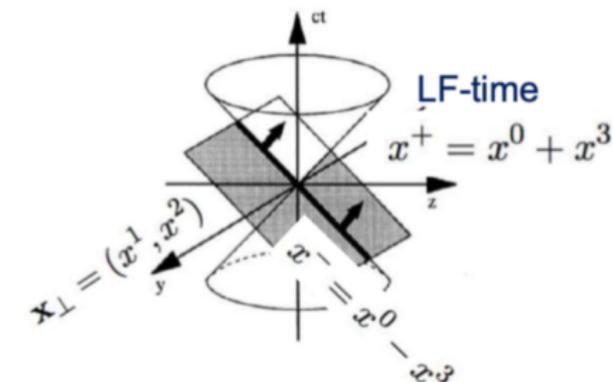
# Valence Proton Double PDF

$$D_3(x_1, x_2; \vec{\eta}_\perp) = \frac{1}{(2\pi)^6} \int d^2k_{1\perp} d^2k_{2\perp} \Psi_3^\dagger(x_1, \vec{k}_{1\perp} + \vec{\eta}_\perp; x_2, \vec{k}_{2\perp} - \vec{\eta}_\perp; x_3, \vec{k}_{3\perp}) \Psi_3(x_1, \vec{k}_{1\perp}; x_2, \vec{k}_{2\perp}; x_3, \vec{k}_{3\perp}).$$

- Fourier transform in  $\vec{\eta}_\perp$ : probability of quarks 1 and 2 for  $x_1$  and  $x_2$  with a separation in the transverse direction  $\vec{y}_\perp$ .
- $D_3 = 0$  for  $x_1 + x_2 > 1$  - momentum conservation. (Below  $\vec{y}_\perp = \vec{0}_\perp$ )



# 3D Hadron Image on the null-plane



The Ioffe-time is useful for studying the relative importance of short and long light-like distances. It is defined as:

$$\tilde{z} = x \cdot P_{target} = x^- P_{target}^+ / 2 \quad \text{on the hyperplane } x^+ = 0$$

Miller & Brodsky, PRC 102, 022201 (2020)

## Ioffe-time image - valence state

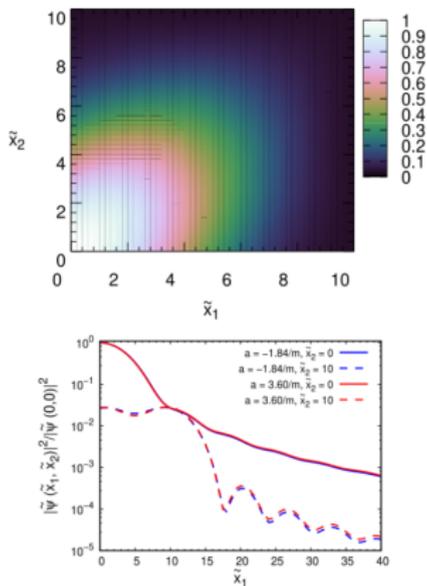
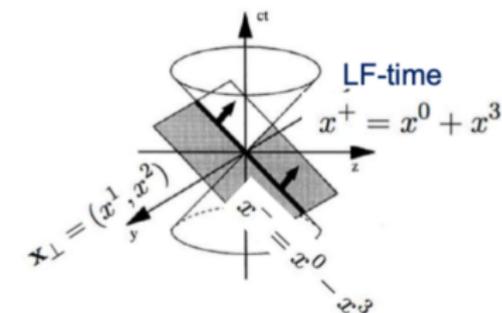
INS FROM ... PHYS. REV. D **104**, 114012 (2021)

FIG. 3. Upper panel: squared modulus of the Ioffe-time distribution as a function of  $\tilde{x}_1$  and  $\tilde{x}_2$ , for the model I. Lower panel: squared modulus of the Ioffe-time distribution as a function of  $\tilde{x}_1$  for two fixed values of  $\tilde{x}_2$ , namely  $\tilde{x}_2 = 0$  (solid line) and  $\tilde{x}_2 = 10$  (dashed line). Results shown for the model I (blue line) and model II (red line).



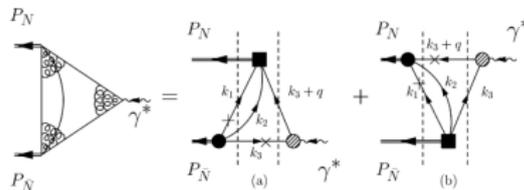
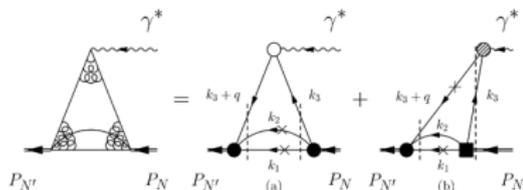
$$\begin{aligned} \Phi(\tilde{x}_1, \tilde{x}_2) &\equiv \tilde{\Psi}_3(\tilde{x}_1, \vec{0}_\perp, \tilde{x}_2, \vec{0}_\perp) \\ &= \int_0^1 dx_1 e^{i\tilde{x}_1 x_1} \int_0^{1-x_1} dx_2 \int_0^1 dx_3 \\ &\quad \times \delta(1 - x_1 - x_2 - x_3) e^{i\tilde{x}_2 x_2} \phi(x_1, x_2, x_3). \end{aligned}$$

## Time-like and space-like nucleon EM factors

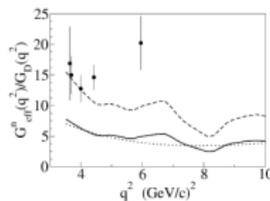
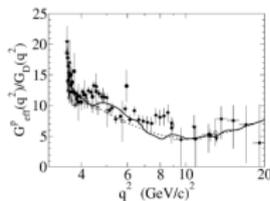
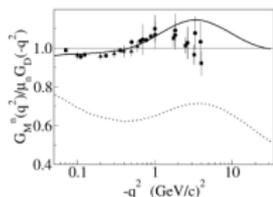
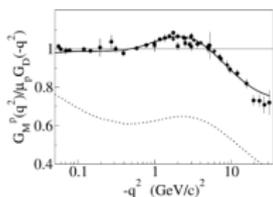
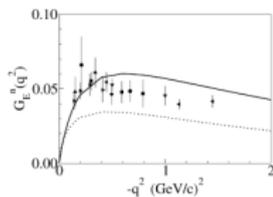
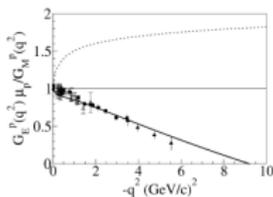
de Melo, TF, Pace, Pisano, Salmè, PLB 671 (2009) 153

## TL nucleon EM FF

J.P.B.C. de Melo et al. / Physics Letters B 671 (2009) 153–157



J.P.B.C. de Melo et al. / Physics Letters B 671 (2009) 153–157

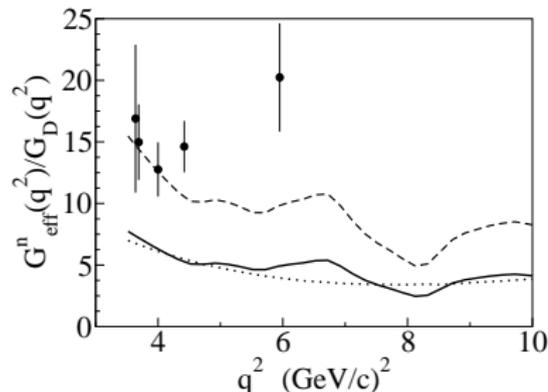
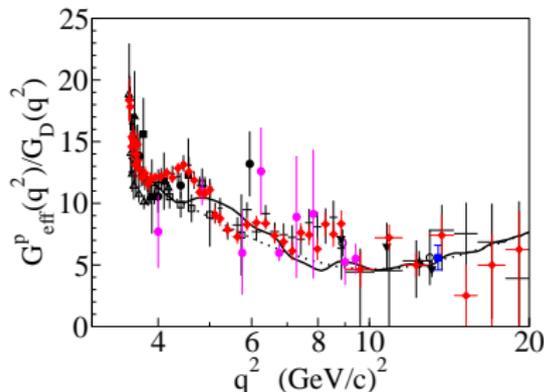


# Proton and Neutron *effective* form factor in the TL region

Thanks to Giovanni Salmè for the SL and TL slides

★ Parameter free result ★

Parameter free like the new evaluation of the SL  $\mu_p G_E^p / G_M^p$

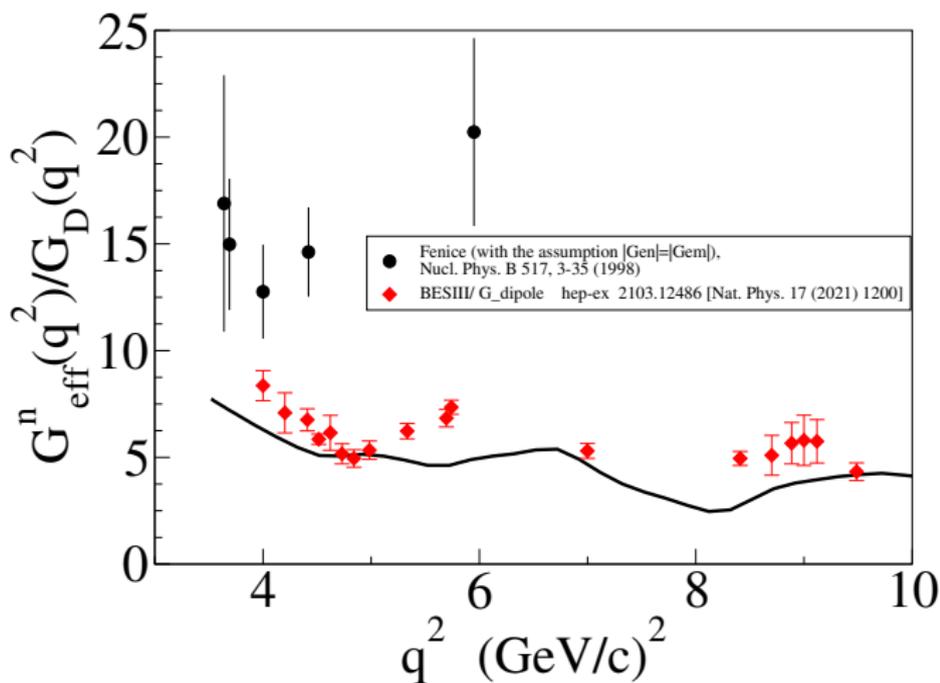


Solid line: full calculation - Dotted line: bare production (no VM).

Proton: Missing strength at  $q^2 = 4.5 \text{ (GeV/c)}^2$  and  $q^2 = 8 \text{ (GeV/c)}^2$

Neutron: Dashed line: solid line arbitrarily  $\times 2$ .

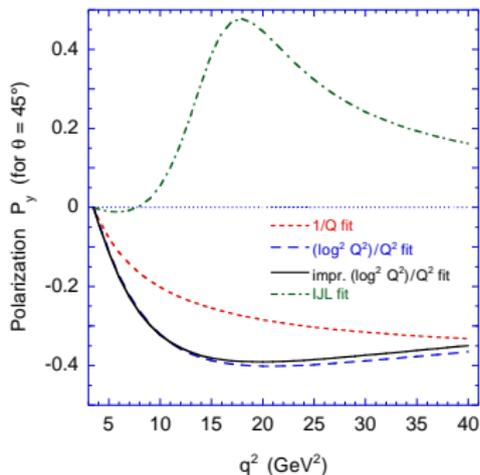
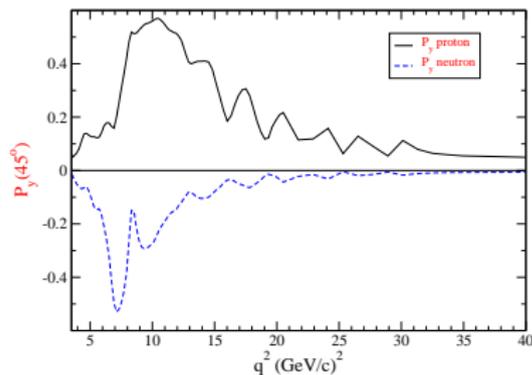
$$G_{eff}(q^2) = \sqrt{\frac{2\tau |G_M(q^2)|^2 + |G_E(q^2)|^2}{2\tau + 1}}$$

Neutron *effective* TL form factor and recent BESIII data

TL proton and neutron polarization orthogonal to the scattering plane: no polarized electron beam !

$$P_y(\theta_{CM}) = -\sin(2\theta_{CM}) \frac{\Im m\{G_E(q^2)G_M^*(q^2)\}}{D\sqrt{\tau}}$$

$$\tau = \frac{q^2}{4M_N^2} \text{ and } D = [1 + \cos^2(\theta_{CM})] |G_M(q^2)|^2 + \sin^2(\theta_{CM}) \frac{|G_E(q^2)|^2}{\tau}$$

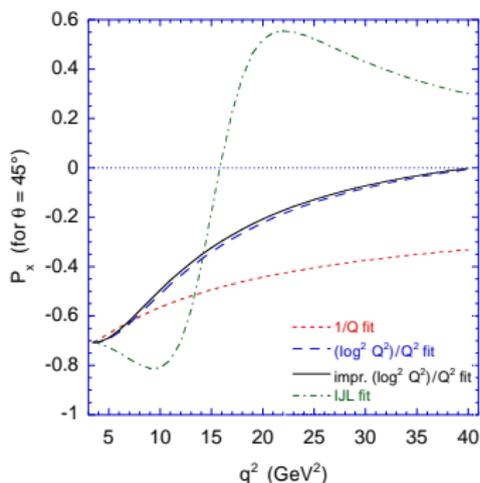
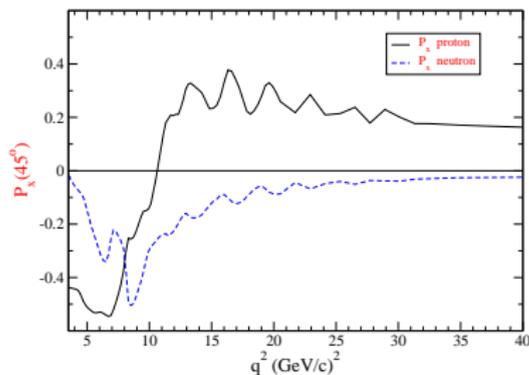


LF Constituent Quark Model

Brodsky, Carlson, Hiller and  
Dae Sung Hwang PRD **69**,  
054022 (2004).

TL proton and neutron polarization orthogonal to incident beams in the scattering plane: polarized electron beam !

$$P_x(\theta_{CM}) = P_e 2\sin(\theta_{CM}) \frac{\Re\{G_E(q^2)G_M^*(q^2)\}}{D\sqrt{T}}$$

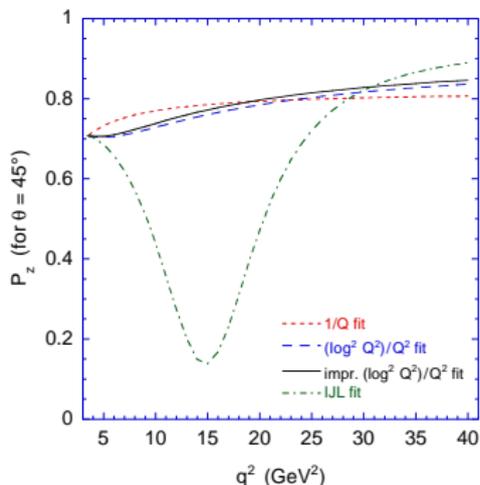
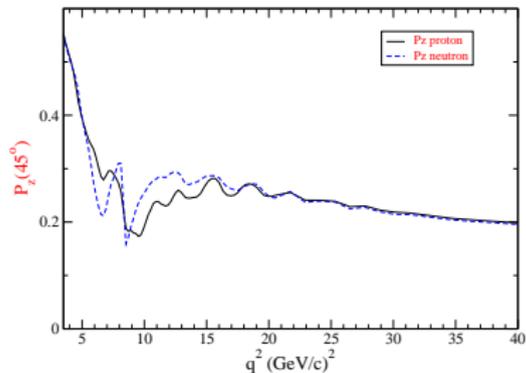


LF Constituent Quark Model

Brodsky, Carlson, Hiller and  
Dae Sung Hwang PRD **69**,  
054022 (2004).

TL proton and neutron polarization along the incident beams: polarized electron beam !

$$P_z(\theta_{CM}) = P_e 2\cos(\theta_{CM}) \frac{|G_M(q^2)|^2}{D}$$



LF Constituent Quark Model

Brodsky, Carlson, Hiller and  
Dae Sung Hwang PRD **69**,  
054022 (2004).

# About the zero $G_E^p/G_M^p$ : BSE in Euclidean space

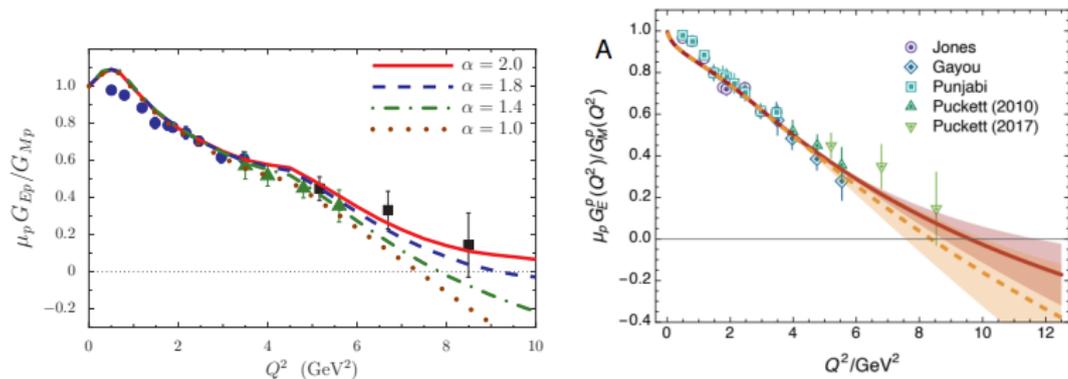


Figure: Left: Cloët, Roberts and Thomas (PRL **111**, 101803 (2013)). Right: Yao, Binosi, Cu and Roberts, [arXiv:2403.08088 [hep-ph]]

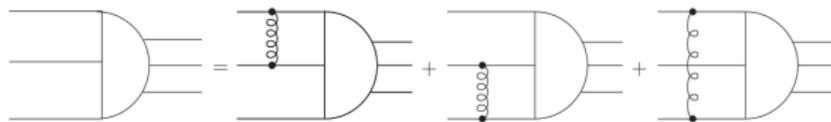
- Cloët, Roberts and Thomas: **quark running mass** and the **position of the zero**.  $\alpha$  : governs the fall-off of the running mass in the intermediate region.
- Yao, Binosi, Cu and Roberts: **improved extrapolation** from  $Q^2 \lesssim 4 \text{ GeV}^2$  to  $Q^2 \gtrsim 12 \text{ GeV}^2$  - statistical Schlessinger point method.  $Q_{G_E^p}^{p, \text{zero}} = 8.86^{+1.93}_{-0.86} \text{ GeV}^2$
- LF Constituent Quark Model:  $Q_{G_E^p}^{p, \text{zero}} = 9 \text{ GeV}^2$

★ **It is desirable to have a Minkowski space version of the nucleon BS model!**

# Light-front projected t'Hooft-like equation for baryons

In preparation Kaur, Mondal, Lan, Zhao, de Melo, TF...

- Bethe-Salpeter equation in 1+1 with the gluon interaction in the LC gauge



$$\Psi_M(k_1, k_2, k_3) = -\frac{i g^2}{(2\pi)^2} S_1(k_1) \otimes S_2(k_2) \otimes S_3(k_3) \\ \otimes \sum_{i=1}^3 S_i^{-1}(k_i) \otimes \int d^2 k'_j \frac{\gamma_j^+ \otimes \gamma_k^+}{(k_j^+ - k'_j)^2} \Psi_M(k_i, k'_j, k'_k)$$

- Inspired in the t'Hooft equation for mesons [NPB75, 461 (1974)]
- LF projection 2-body BSE - eliminate the LF time with quasi potential approach [Sales et al PRC61, 044003 (2000)]
- LF projection 3-body BSE with QP approach [Marinho,TF, PoS LC2008 (2008) 036; Guimarães etal JHEP 08 (2014); Ydrefors,TF PRD104 (2021)114012]

- LO three-quark LF-equation: **valence truncation** [Bars, PRL36, 1521 (1976)]

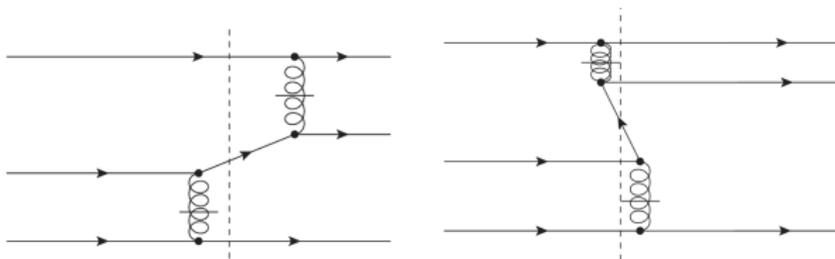


Figure: Intermediate states:  $|3q\rangle$  (included in LO) and  $|4q\bar{q}\rangle$  (excluded in LO).

- UV:  $\phi([x_i]) \simeq (x_1 x_2 x_3)^\beta$  t'Hooft ansatz  $\frac{\pi}{2} \frac{m^2}{g^2} - 1 + \pi\beta \cot \pi\beta = 0$

## Results: 3-quark spectrum and DDA in QCD 1+1

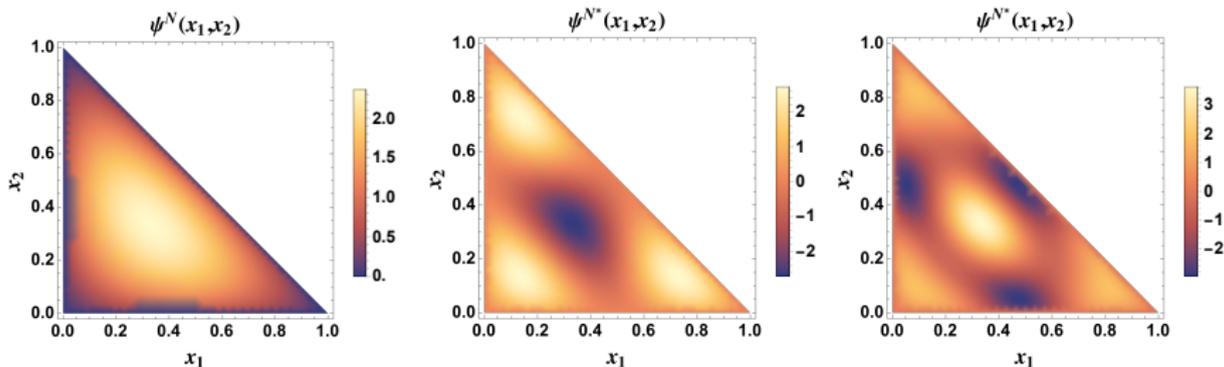
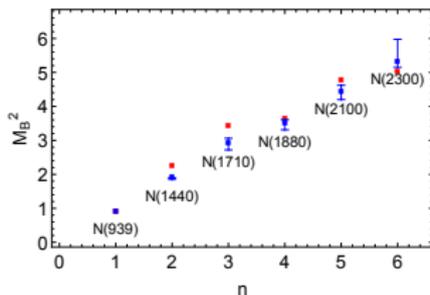
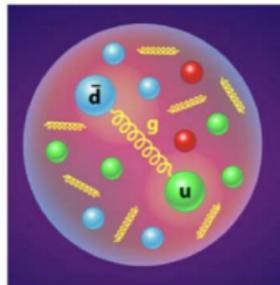
Preliminary calculations:  $m = 210$  MeV and  $g = 330$  MeV

Figure: DDA: N(939) (left), N(1440) (middle) and N(1710) (right).

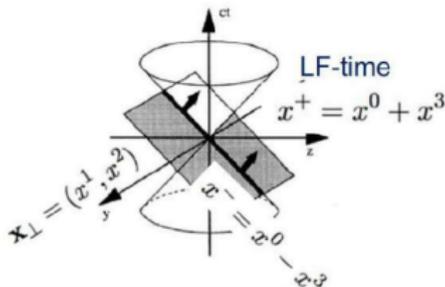
# Pion - Interesting?

## Pions

- Pion is the **Goldstone boson** associated with spontaneous symmetry breaking of chiral  $SU(2)_L \times SU(2)_R$  symmetry
- **Lightest hadron**
- Made up of  $q$  and  $\bar{q}$  constituents



## Light-front hypersurface



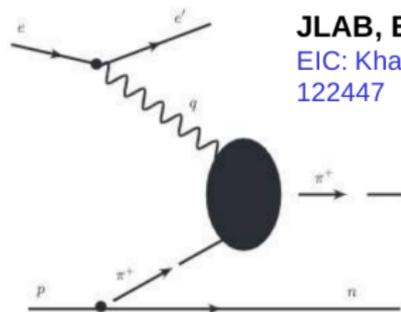
barryp@jlab.org

Credits to Patrick Barry

3

$$|\pi\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle + |q\bar{q}2g\rangle + \dots$$

# How to look?



**JLAB, EIC...**

EIC: Khalek et al. NPA 1026 (2022)  
122447

FIG. 1. Sullivan process:  $ep \rightarrow e'\pi^+n$  scattering. The black blob represents the half-on-mass shell photo absorption amplitude. Diagrammatic representation of the pion pole amplitude for  $p(e, e')\pi^+n$  process.

off-shell pion EM FF: Choi, TF, Ji, de Melo, PRD 100, 116020 (2019)

Leño, de Melo, TF, Choi, Ji, PRD 110, 074035 (2024)

## Dressed quarks/gluons?

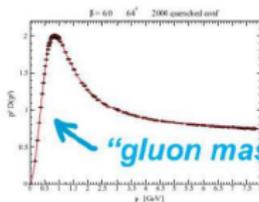
Haag theorem!!!!

6

INPUTS FROM LQCD in Landau gauge: SL momenta

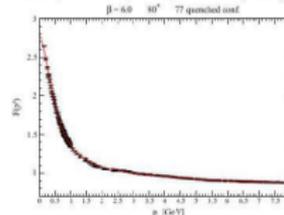
Gluon propagator

$$D_{\mu\nu}^{ab}(q) = -i\delta^{ab} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) D(q^2)$$

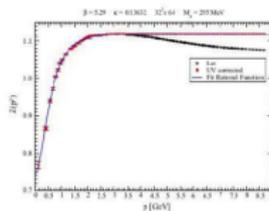
Dudal, Oliveira, Silva, Ann. Phys. **397**, 351 (2018)

Ghost propagator

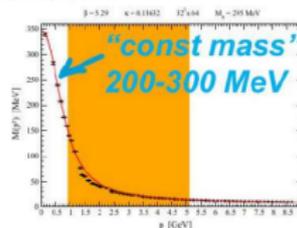
$$D_{gh}(p^2) = \frac{F(p^2)}{p^2}$$

Duarte, Oliveira, Silva, PRD **94** (2016) 014502

Quark propagator

Oliveira, Silva, Skullerud and Sternbeck, PRD **99** (2019) 094506

$$i Z(p^2) \frac{\not{p} + M(p^2)}{p^2 - M^2(p^2)}$$

Parametrizations summarized in Oliveira, de Paula, Frederico, de Melo, EPJ C **79** (2019) 116

17

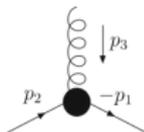
# The Quark-Gap Equation and the Quark-Gluon Vertex

Spontaneous Chiral symmetry breaking & pion as a Goldstone boson  
(origin of the nucleon mass – “constituent quarks”, Roberts, Maris, Tandy, Cloet, Maris...)

Schwinger-Dyson eq.  
Quark propagator



Quark-gluon vertex



$$\Gamma_{\mu}^a(p_1, p_2, p_3) = g t^a \Gamma_{\mu}(p_1, p_2, p_3)$$

$$\Gamma_{\mu}(p_1, p_2, p_3) = \Gamma_{\mu}^{(L)}(p_1, p_2, p_3) + \Gamma_{\mu}^{(T)}(p_1, p_2, p_3)$$

Longitudinal component

$$\Gamma_{\mu}^L(p_1, p_2, p_3) = -i \left( \lambda_1 \gamma_{\mu} + \lambda_2 (\not{p}_1 - \not{p}_2) (p_1 - p_2)_{\mu} + \lambda_3 (p_1 - p_2)_{\mu} + \lambda_4 \sigma_{\mu\nu} (p_1 - p_2)^{\nu} \right)$$

Rojas, de Melo, El-Bennich, Oliveira, Frederico, JHEP 1310 (2013) 193;

Oliveira, Paula, Frederico, de Melo EPJC **78**(7), 553 (2018) & EPJC 79 (2019) 116

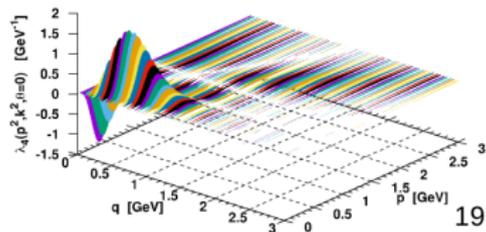
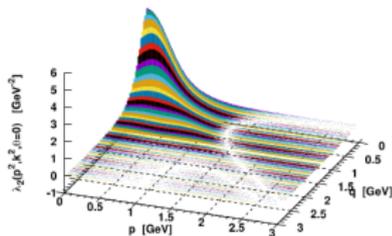
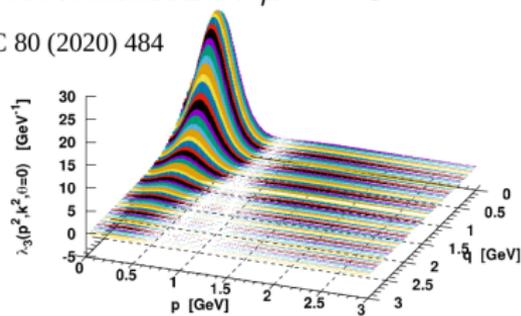
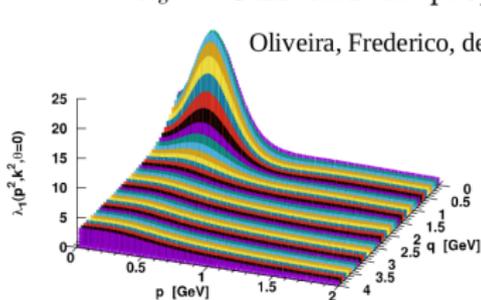
Oliveira, Frederico, de Paula, EPJC 80 (2020) 484

## quark-gluon vertex from factors

- Schwinger-Dyson eq. quark self-energy
- Longitudinal components quark-gluon vertex
- Slanov-Taylor identity & Quark-Ghost Kernel
- Padé approximants
- Error minimization  $\sim 2-4\%$

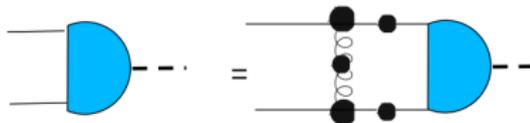
$\alpha_s = 0.22$  and all propagators renormalised at  $\mu = 4.3\text{GeV}$

Oliveira, Frederico, de Paula, EPJC 80 (2020) 484



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## How we model: BSE quark-antiquark & pion model



*Ladder approximation (L): suppression of XL for  $N_c=3$  in a bosonic system  
[A. Nogueira, CR Ji, Ydrefors, TF, PLB777(2017) 207]*

- dressed quark propagator (mass =255MeV)  $S(P) = \frac{i}{\not{P} - m + i\epsilon}$
- dressed gluon propagator (mass =637MeV)  $i\mathcal{K}_V^{(Ld)\mu\nu}(k, k') = -ig^2 \frac{g^{\mu\nu}}{(k - k')^2 - \mu^2 + i\epsilon}$
- dressed quark-gluon vertex (306 MeV)  $\lambda_1 \gamma_\mu F(q) = \frac{\mu^2 - \Lambda^2}{q^2 - \Lambda^2 + i\epsilon}$
- Model parameters: quark and gluon masses & quark-gluon vertex

SOLUTION IN MINKOWSKI SPACE [pion mass  $\rightarrow$  g]

## Pion BS amplitude

$$\Phi(k, p) = S_1 \phi_1 + S_2 \phi_2 + S_3 \phi_3 + S_4 \phi_4$$

$$S_1 = \gamma_5 \quad S_2 = \frac{1}{M} \not{p} \gamma_5 \quad S_3 = \frac{k \cdot p}{M^3} \not{p} \gamma_5 - \frac{1}{M} \not{k} \gamma_5 \quad S_4 = \frac{i}{M^2} \sigma_{\mu\nu} p^\mu k^\nu \gamma_5$$

## Main Tool: Nakanishi Integral Representation (NIR)

(Nakanishi 1962)

Each BS amplitude component:

$$\Phi_i(k, p) = \int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g_i(\gamma', z')}{(\gamma' + \kappa^2 - k^2 - p \cdot kz' - i\epsilon)^3} \quad \kappa^2 = m^2 - \frac{M^2}{4}$$

**Bosons:** Kusaka and Williams, PRD 51 (1995) 7026;

**Light-front projection: integration in  $k$**

Carbonell&Karmanov EPJA27(2006)1;EPJA27(2006)11;

TF, Salme, Viviani PRD89(2014) 016010,...

**Fermions (0<sup>-</sup>):** Carbonell and Karmanov EPJA 46 (2010) 387;

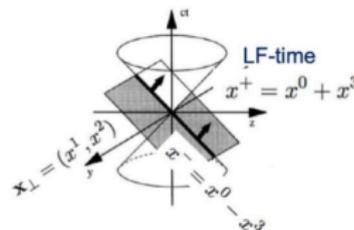
de Paula, TF, Salmè, Viviani PRD 94 (2016) 071901;

de Paula, TF, Pimentel, Salmè, Viviani, EPJC 77 (2017) 764

# Projecting BSE onto the LF hyper-plane $x^+=0$

## LF amplitudes

$$\psi_i(\gamma, \xi) = \int \frac{dk^-}{2\pi} \phi_i(k, p) = -\frac{i}{M} \int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{[\gamma + \gamma' + m^2 z^2 + (1-z^2)\kappa^2]^2}$$



$$\int_0^\infty d\gamma' \frac{g_i(\gamma', z')}{[\gamma + \gamma' + m^2 z^2 + (1-z^2)\kappa^2]^2} = iMg^2 \sum_j \int_0^\infty d\gamma' \int_{-1}^1 dz' \mathcal{L}_{ij}(\gamma, z; \gamma' z') g_j(\gamma, z')$$

**Generalized Stieltjes transform: invertible** Carbonell, TF, Karmanov PLB769 (2017) 418

*The coupled equations are formally equivalent to BSE, once NIR is applied, and the validity of NIR is assessed by the existence of unique solutions to the GEVP!*

**Kernel contains singular contributions!**

- Kernel of the LF projected pion BSE with NIR  
de Paula, TF, Salmè, Viviani PRD 94 (2016) 071901;  
de Paula, TF, Pimentel, Salmè, Viviani, EPJC 77 (2017) 764
- end-point singularities in the  $k^-$  integration (zero-modes)

T.M. Yan , Phys. Rev. D **7**, 1780 (1973)

$$\mathcal{I}(\beta, y) = \int_{-\infty}^{\infty} \frac{dx}{[\beta x - y \mp i\epsilon]^2} = \pm \frac{2\pi i \delta(\beta)}{[-y \mp i\epsilon]}$$

Kernel with delta and its derivative!

End-point singularities—more intuitive: can be treated by the pole-dislocation method  
de Melo et al. NPA631 (1998) 574C, PLB708 (2012) 87

## Results

- Pion valence wf,  $f_\pi$  and DA
- Pion image
- EM FF
- PDF
- TMDs

# BS norm, valence wave function, decay constant

Paula, Ydrefors, Alvarenga Nogueira, TF and Salme PRD 103 014002 (2021).

**Normalization:**  $i N_c \int \frac{d^4 k}{(2\pi)^4} [\phi_1 \phi_1 + \phi_2 \phi_2 + b \phi_3 \phi_3 + b \phi_4 \phi_4 - 4 b \phi_1 \phi_4 - 4 \frac{m}{M} \phi_2 \phi_1] = -1$

**Valence wf:**

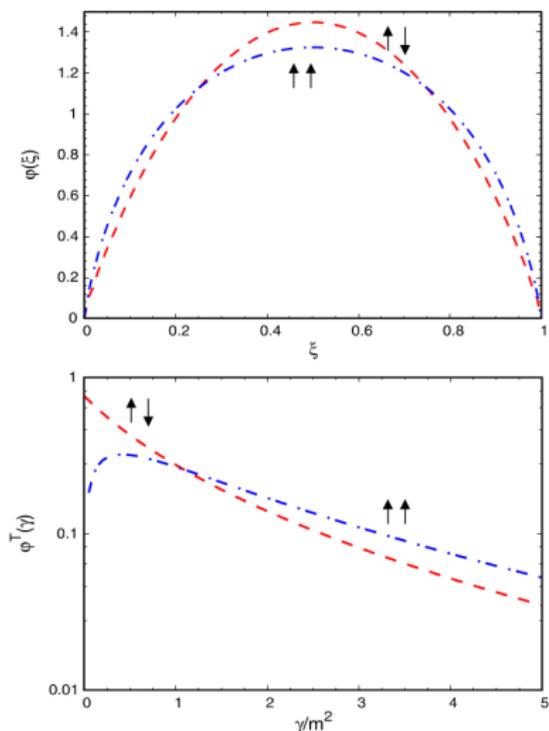
$$\left\{ \begin{array}{l} \psi_{\uparrow\downarrow}(\gamma, z) = -i \frac{M}{4p^+} \int \frac{dk^-}{2\pi} \text{Tr}[\gamma^+ \gamma_5 \Phi(k; p)] \\ \qquad \qquad \qquad = \psi_2(\gamma, z) + \frac{z}{2} \psi_3(\gamma, z) + \int_0^\infty \frac{d\gamma'}{M^3} \frac{\partial g_3(\gamma', z)/\partial z}{[\gamma + \gamma' + z^2 m^2 + (1 - z^2) \kappa^2]} \\ \psi_{\uparrow\uparrow}(\gamma, z) = \frac{\sqrt{\gamma} M}{4ip^+} \int \frac{dk^-}{2\pi} \text{Tr}[\sigma^{+i} \gamma_5 \Phi(k; p)] = \frac{\sqrt{\gamma}}{M} \psi_4(\gamma, z) \end{array} \right. \quad \gamma = k_\perp^2 \text{ and } z = 2\xi - 1$$

*Aligned spin component of Purely relativistic nature!*

**Valence probability:**  $P_{\text{val}} = \frac{N_c}{16\pi^2} \int_{-1}^1 dz \int_0^\infty d\gamma [|\psi^{\uparrow\downarrow}(\gamma, z)|^2 + |\psi^{\uparrow\uparrow}(\gamma, z)|^2]$

**Decay constant:**  $f_\pi = -i \frac{N_c}{p^+} \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[\gamma^+ \gamma_5 \Phi(p, k)] = \frac{2 N_c}{M} \int \frac{d^2 k_\perp dk^+}{(2\pi)^2 2\pi} \psi_{\uparrow\downarrow}(\gamma, z)$   
 $= 130 \text{ MeV}$       The experimental value of  $f_\pi$  is  $130.50 \pm 0.017 \text{ MeV}$

W. DE PAULA *et al.* Valence Distribution and transverse amplitudes



Prob\_val=0.7

Prob\_antialigned=0.57

Prob\_aligned=0.13

$$\varphi_{\uparrow\downarrow}(\xi) = \frac{\int_0^\infty d\gamma \psi_{\uparrow\downarrow}(\gamma, z)}{\int_0^1 d\xi \int_0^\infty d\gamma \psi_{\uparrow\downarrow}(\gamma, z)},$$

$$\varphi_{\uparrow\uparrow}(\xi) = \frac{\int_0^\infty d\gamma \psi_{\uparrow\uparrow}(\gamma, z)}{\int_0^1 d\xi \int_0^\infty d\gamma \psi_{\uparrow\uparrow}(\gamma, z)}.$$

$$\varphi_{\uparrow\downarrow}^T(\gamma) = \frac{\int_0^1 d\xi \psi_{\uparrow\downarrow}(\gamma, z)}{\int_0^1 d\xi \int_0^\infty d\gamma \psi_{\uparrow\downarrow}(\gamma, z)},$$

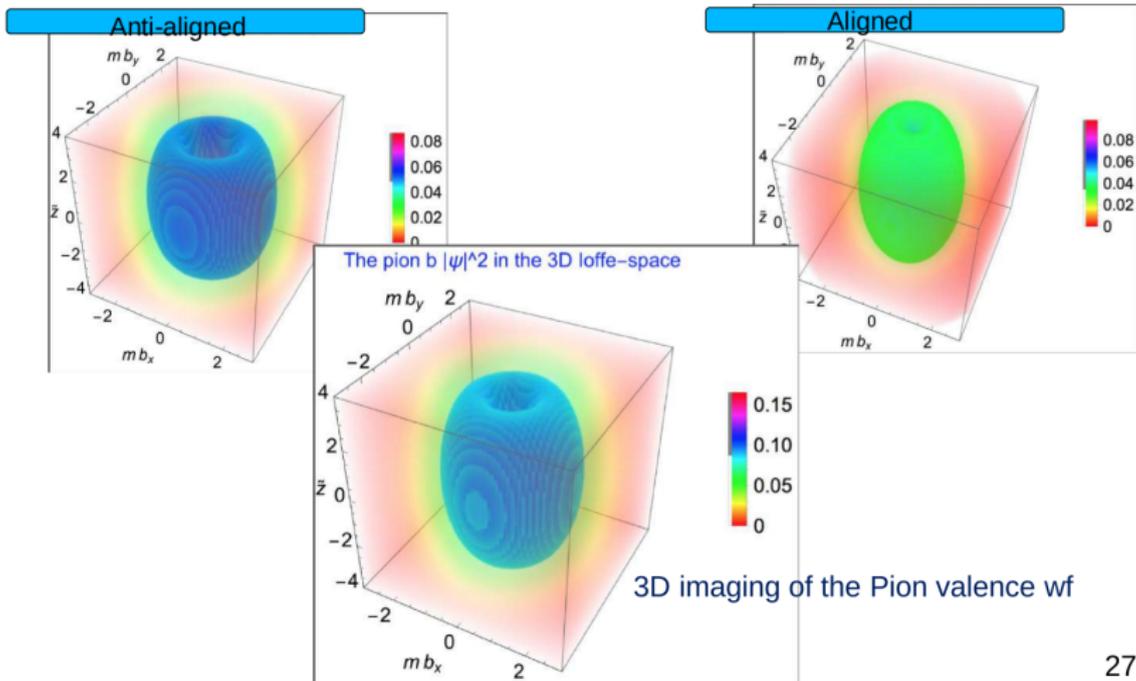
$$\varphi_{\uparrow\uparrow}^T(\gamma) = \frac{\int_0^1 d\xi \psi_{\uparrow\uparrow}(\gamma, z)}{\int_0^1 d\xi \int_0^\infty d\gamma \psi_{\uparrow\uparrow}(\gamma, z)},$$

Transverse amplitude can be computed directly in Euclidean and Minkowski spaces!

Gutierrez et al PLB 759 (2016) 131

# 3D Pion image on the null-plane: Spin configurations

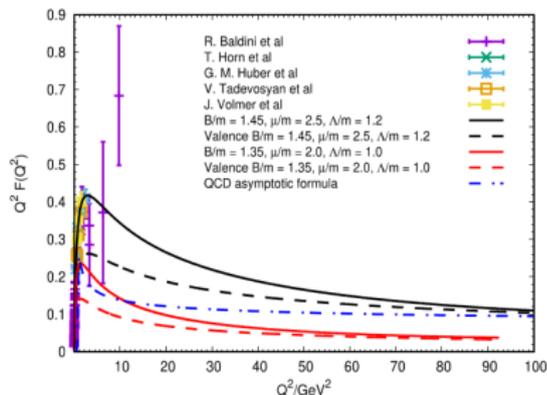
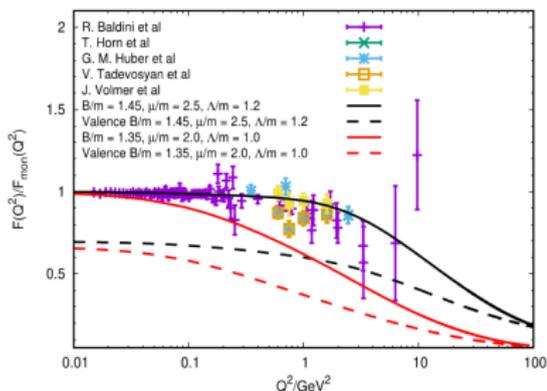
Space-time structure of the pion in terms of  $z = x^- p^+ / 2$  and transverse coord.  $\{b_x, b_y\}$



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# Pion EM Form Factor

Alvarenga Nogueira, de Paula, TF, Ydrefors, Salmè, PLB 820, 136494 (2021)



$$Q^2 F_{\text{asyp}}(Q^2) = 8\pi\alpha_s(Q^2)f_\pi^2$$

G. Lepage, S. J. Brodsky Phys. Lett. B 87 (1979) 359

## Decomposition of the pion EM form factor

$$F_{\pi}(Q^2) = \sum_n F_n(Q^2) = F_{val}(Q^2) + F_{nval}(Q^2)$$

qq+gluons

$$r_{\pi}^2 = P_{val} r_{val}^2 + (1 - P_{val}) r_{nval}^2$$

$r_{\pi}$ (fm)	$r_{val}$ (fm)	$r_{nval}$ (fm)
0.663	0.710	0.538

$0.657 \pm 0.003$  fm    B. Ananthanarayan, I. Caprini, D. Das, Phys. Rev. Lett. 119 (2017) 132002

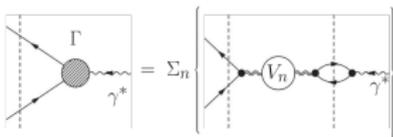
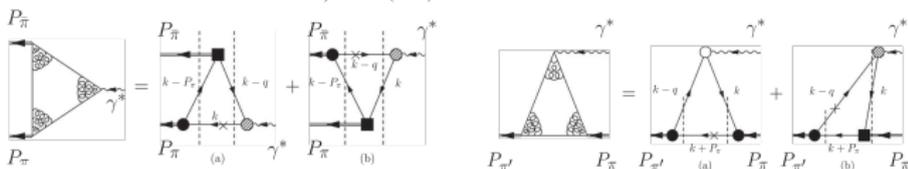
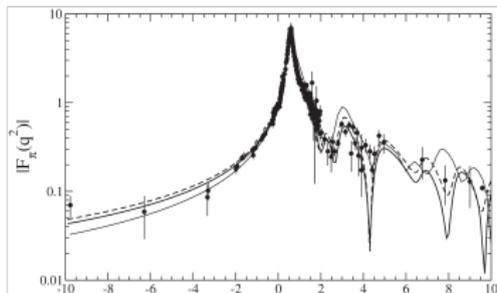
**higher Fock-components** → **large virtuality** → **more compact**

Kharzeev, "Mass radius of the proton" PRD104, 054015 (2021)

$$R_m = 0.55 \pm 0.03 \text{ fm} \quad R_C = 0.8409 \pm 0.0004 \text{ fm}$$

# Quick view: Pion SL and TL FF [PRD 73, 074013 (2006)]

$q^+ = q^0 + q^3 > 0$  Exp.: Baldini et al., EPJC 11,709(1999); NPA666, 38(2000); (priv comm); Volmer et al., PRL86,1713(2001).



## Comparison with experimental data

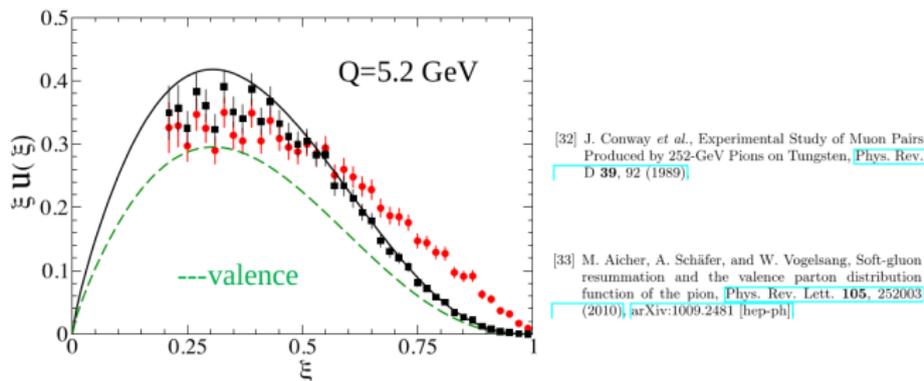


FIG. 2. (Color online). The distribution function  $\xi u(\xi)$  in a pion. Solid line: full calculation (see Eqs. (7) and (8)), obtained from the BS amplitude solution of the BSE with  $m = 255$  MeV,  $\mu = 637.5$  MeV and  $\Lambda = 306$  MeV, and evolved from the initial scale  $Q_0 = 0.360$  GeV to  $Q = 5.2$  GeV (see text). Dashed line: the evolved LF valence component, Eq. (9). Full dots: experimental data from Ref. [32]. Full squares: reanalyzed data by using the ratio between the fit 3 of Ref. [33], evolved to 5.2 GeV, and the experimental data [32], at each data point, so that the resummation effects (see text) are accounted for.

# Pion: Quark unpolarized transverse-momentum distribution functions

de Paula, TF, Salmè, EPJC 83 (2023) 985

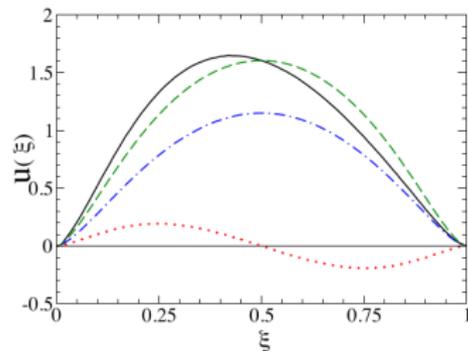
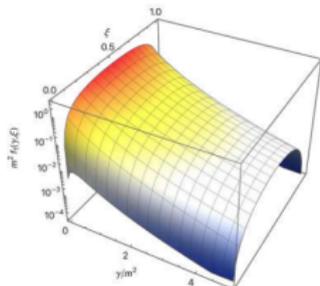
T-even uTMD leading-twist from the quark-quark correlator

Mulders & Tangerman NPB461, 197 (1996)

$$f_1^q(\gamma, \xi) = \frac{N_c}{4} \int d\phi_{\mathbf{k}_\perp} \int_{-\infty}^{\infty} \frac{dy^- d\mathbf{y}_\perp}{2(2\pi)^3} \\ \times e^{i[\xi P^+ \frac{y^-}{2} - \mathbf{k}_\perp \cdot \mathbf{y}_\perp]} \langle P | \bar{\psi}_q(-\frac{y^-}{2}) \gamma^+ \psi_q(\frac{y^-}{2}) | P \rangle \Big|_{y^+=0}$$

$$\gamma = |\mathbf{k}_\perp|^2$$

$$f_1^q(\gamma, \xi) = \frac{N_c}{4(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk^+}{2(2\pi)} \delta\left(k^+ + \frac{P^+}{2} - \xi P^+\right) \\ \times \int_{-\infty}^{\infty} dk^- \int_0^{2\pi} d\phi_{\mathbf{k}_\perp} \text{Tr} [S^{-1}(-p_{\bar{q}}) \bar{\Phi}(k, P) \gamma^+ \Phi(k, P)]$$

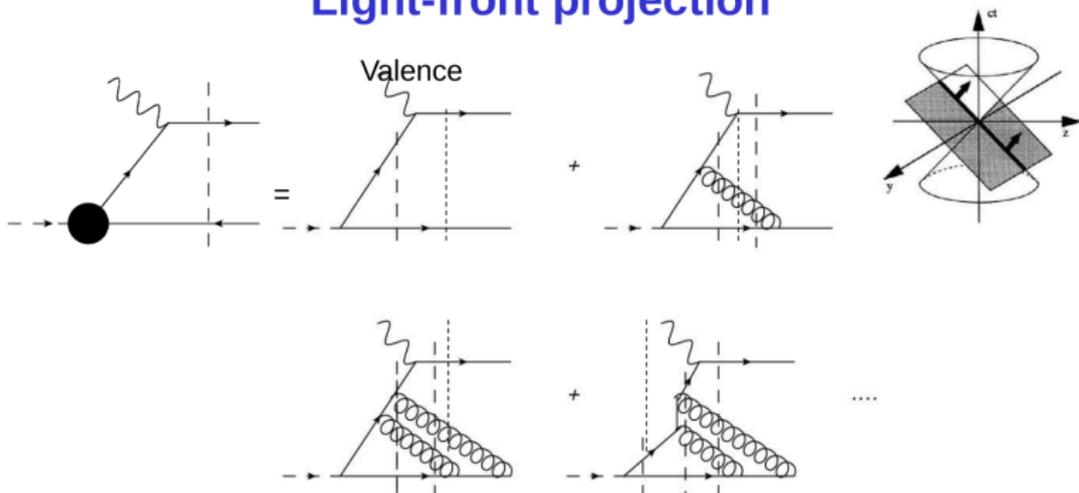


$$\langle \xi_q \rangle = \int_0^1 d\xi \int_0^\infty d\gamma \xi f_1^q(\gamma, \xi) = 0.471$$

$$|\pi\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle + |q\bar{q}2g\rangle + \dots$$

# Bethe-Salpeter amplitude: beyond the valence states

## Light-front projection



- higher Fock-components  $|\pi\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle + |q\bar{q}2g\rangle + \dots$
- gluon radiation from initial state interaction (ISI)
- 

Sales, TF, Carlson, Sauer, PRC 63, 064003 (2001);

Marinho, TF, Pace, Salme, Sauer, PRD 77, 116010(2008)

## Gluon momentum in the pion

$$|\pi\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle + |q\bar{q}2g\rangle + \dots$$

quark momentum distribution

$$u^q(\xi) = \sum_{n=2}^{\infty} \left\{ \prod_i^n \int \frac{d^2 k_{i\perp}}{(2\pi)^2} \int_0^1 d\xi_i \right\} \\ \times \delta(\xi - \xi_1) \delta\left(1 - \sum_{i=1}^n \xi_i\right) \delta\left(\sum_{i=1}^n \mathbf{k}_{i\perp}\right) \\ \times |\Psi_n(\xi_1, \mathbf{k}_{1\perp}, \xi_2, \mathbf{k}_{2\perp}, \dots)|^2,$$

first-moment

$$\langle \xi_q \rangle = P_{val} \langle \xi_q \rangle_{val} + \sum_{n>2} P_n \langle \xi_q \rangle_n \\ \mathbf{0.471} \quad \mathbf{0.5} \\ = P_{val} \langle \xi_q \rangle_{val} + (1 - P_{val}) \langle \xi_q \rangle_{HFS} \\ \mathbf{P_{val}=0.7} \quad \mathbf{0.4}$$

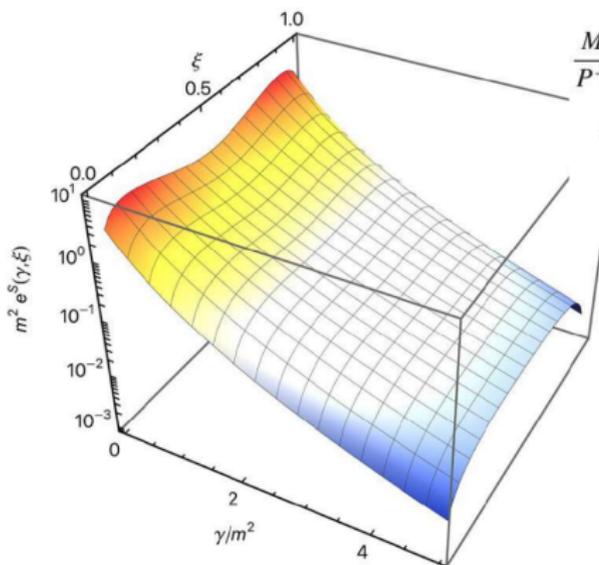
momentum sum-rule in the HFS

$$\langle \xi_q \rangle_{HFS} = 1 - \langle \xi_{\bar{q}} \rangle_{HFS} - \langle \xi_g \rangle \\ \mathbf{0.2}$$

**Glucos carry 6% of the longitudinal momentum of the pion!**

@ the pion scale

## Subleading-twist 3 uTMDs



$$\begin{aligned} & \frac{M}{P^+} e^q(\gamma, \xi) \\ &= \frac{N_c}{4} \int_0^{2\pi} d\phi_{\hat{\mathbf{k}}_\perp} \int_{-\infty}^{\infty} \frac{dy^- dy_\perp}{2(2\pi)^3} \\ & \times e^{i[\xi P^+ \frac{y^-}{2} - \mathbf{k}_\perp \cdot \mathbf{y}_\perp]} \langle P | \bar{\psi}_q(-\frac{y}{2}) \mathbb{1} \psi_q(\frac{y}{2}) | P \rangle \Big|_{y^+=0}. \end{aligned}$$

de Paula, TF, Salmè, EPJC 83 (2023)  
985

**Fig. 7** Pion unpolarized transverse-momentum distribution  $e^S(\gamma, \xi)$ , Eq. (44), at the initial scale

# Pion PDF: Gluonic contributions BLFQ & BSE

Lan, Mondal, Zhao, TF, Vary, PRD 111, L111903 (2025)

$$P^- P^+ |\Psi\rangle = M^2 |\Psi\rangle, \quad |\Psi\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle \dots$$

$$\psi_{q\bar{q}g}^{(s_q, s_{\bar{q}}, \lambda)}(i_q, i_{\bar{q}}, a) = 1/(M_\pi^2 - M_{0, q\bar{q}g}^2) [V \psi_{q\bar{q}}^{(s_q, s_{\bar{q}})}(i_q, i_{\bar{q}})]$$

BLFQ: H.O. Confinement + coupling of  $|q\bar{q}\rangle$  and  $|q\bar{q}g\rangle$  states

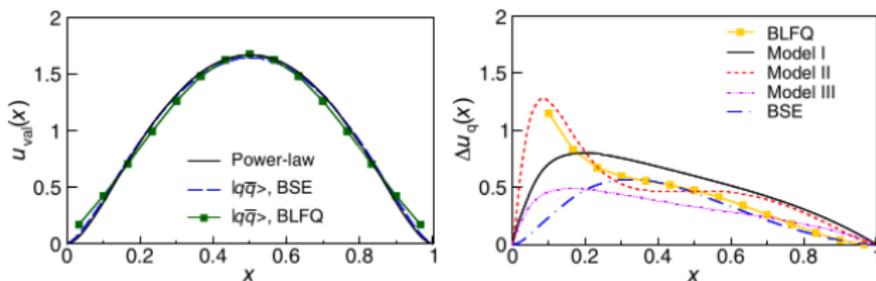


TABLE I. Quark and gluon masses (GeV) for three parameter sets, with  $q\bar{q}g$  probabilities.

Model	$m_q$ (GeV)	$m_f$ (GeV)	$m_g$ (GeV)	$1-P_{val}$
I	0.390	0.390	0.600	0.508
II	0.390	5.69	0.600	0.508
III	0.255	0.255	0.638	0.300

# Evolved Pion PDF: Gluonic contributions

PHYS. REV. D **111**, L111903 (2025)

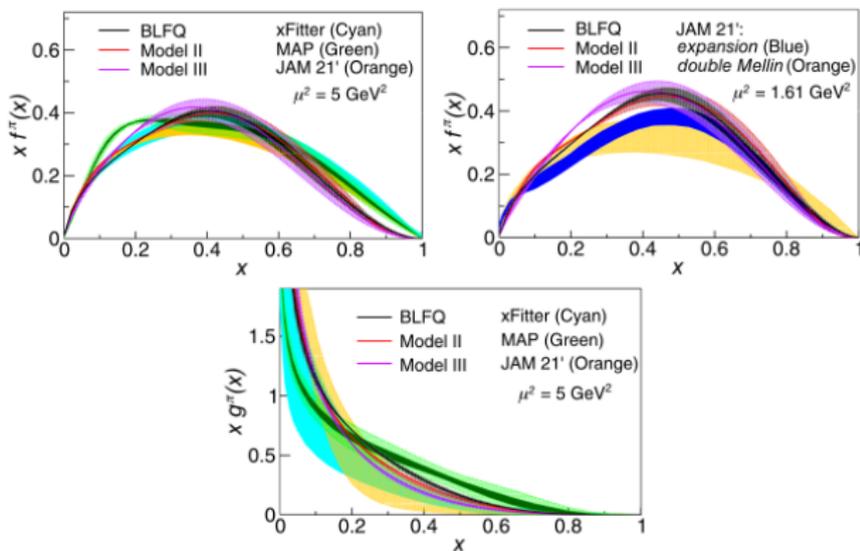


Figure: Initial scales obtained by matching the first moment  $\langle x \rangle_{val} = 0.48 \pm 0.01$  at  $5 \text{ GeV}^2$  [Barry, Sato, Melnitchouk, Ji, PRL121, 152001 (2018)]:

$$\mu_{BLFQ}^2 = 0.34 \pm 0.03 \text{ GeV}^2, \mu_{II0}^2 = 0.30 \pm 0.03 \text{ GeV}^2, \mu_{III0}^2 = 0.26 \pm 0.03 \text{ GeV}^2$$

## Towards Dynamical Chiral symmetry breaking in Minkowski

### Dynamically Dressed Quarks

## Dressing the Quark: Schwinger-Dyson equation

The model: Bare vertices, massive vector boson, Pauli-Villars regulator

Credits to Wayne de Paula

The rainbow ladder Schwinger-Dyson equation in **Minkowski space** is:

$$S_q^{-1}(k) = \not{k} - m_B + ig^2 \int \frac{d^4 q}{(2\pi)^4} \Gamma_\mu(q, k) S_q(k - q) \gamma_\nu D^{\mu\nu}(q),$$

where  $m_B$  is the **quark bare mass** and  $g$  is the coupling constant.

The massive gauge boson is given by

$$D^{\mu\nu}(q) = \frac{1}{q^2 - m_g^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(1 - \xi)q^\mu q^\nu}{q^2 - \xi m_g^2 + i\epsilon} \right],$$

where we have introduced an effective gluon mass  $m_g$ , as suggested by LQCD calculations (see *Dudal, Oliveira and Silva, PRD 89 (2014) 014010*).

The dressed fermion propagator is

$$S_q(k) = \left[ \not{k} A(k^2) - B(k^2) + i\epsilon \right]^{-1}.$$

Duarte, TF, de Paula, Ydrefors, PRD105, 114055 (2022)

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## Schwinger-Dyson equation in Rainbow ladder truncation

The vector and scalar self-energies are given by the NIR, respectively as:

$$A(k^2) = 1 + \int_0^\infty ds \frac{\rho_A(s)}{k^2 - s + i\epsilon},$$

$$B(k^2) = m_B + \int_0^\infty ds \frac{\rho_B(s)}{k^2 - s + i\epsilon}.$$

The quark propagator can also be written as:

$$S_q(k) = R \frac{\not{k} + \bar{m}_0}{k^2 - \bar{m}_0^2 + i\epsilon} + \not{k} \int_0^\infty ds \frac{\rho_V(s)}{k^2 - s + i\epsilon} + \int_0^\infty ds \frac{\rho_S(s)}{k^2 - s + i\epsilon},$$

where  $\bar{m}_0$  is the renormalized mass.

$$\not{k}A(k^2) - B(k^2) = ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q)\gamma_\nu}{q^2 - m_g^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(1-\xi)q^\mu q^\nu}{q^2 - \xi m_g^2 + i\epsilon} \right]$$

 Gauge fixing

$$- i g^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q)\gamma_\nu}{q^2 - \Lambda^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{(1-\xi)q^\mu q^\nu}{q^2 - \xi \Lambda^2 + i\epsilon} \right]$$

 Pauli-Villars regulator

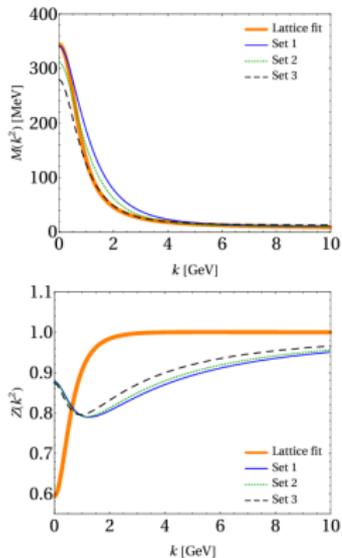


FIG. 1. Landau gauge results for the running mass  $M(k^2)$  and quark wave function  $Z(k^2)$  as functions of spacelike momentum  $k$ , using the sets of parameters given in Table I. Solid thick curves are the fit of LQCD calculations for the mass function and wave function renormalization given in [4].

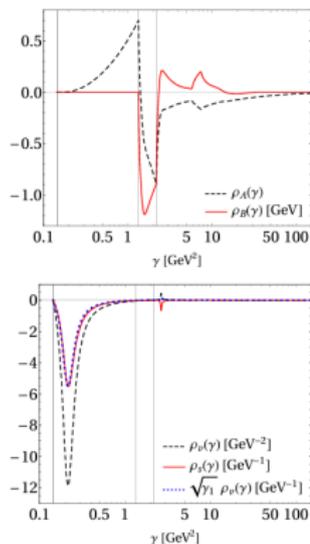


FIG. 2. Spectral densities for the self-energy (upper panel) and for the propagator (lower panel) as functions of  $\gamma$ , computed for set 2 from Table I. The vertical lines are  $\bar{m}_0^2$ ,  $(\bar{m}_0 + m_q)^2$  and  $(\bar{m}_0 + \Lambda)^2$  from the thresholds of the driving terms in Eqs. (12) and (13). ( $\gamma_1 = 0.216 \text{ GeV}^2$ , see text).

Set	$\bar{m}_0$ (GeV)	$m_q$ (GeV)	$\Lambda$ (GeV)	$\alpha$
1	0.42	0.84	1.20	19.70
2	0.38	0.78	1.10	20.30
3	0.35	0.60	1.00	13.25

Set (outputs)	$m_B$ (MeV)	$R$
1	9.29	2.22
2	8.78	2.09
3	11.92	2.64

# Summary and Prospects

- BSE in Minkowski space: proton and pion
  - PDFs, EM FF, TMDs, Ioffe-time Image

- Quark Dressing in Minkowski space

## Prospects:

- K, D, B,  $\rho$ , Nucleon (spin)...
- T-odd TMDs, GTMDs (SL & TL), GFF...
- dressed constituents in BS equation - [Castro et al PLB845 \(2023\) 138159](#)
- Gluon exchange & dressed vertices
- Integral representation to solve the FBS equation
- Expand the applicability of Quantum algorithm for solving the pion BS & FBS equations [Fornetti, et al, PRD 110 \(2024\) 056012](#)
- Confinement (QCD 1+1 t'Hooft model for three-quarks)...

# Acknowledgements

## CAS President's International Fellowship Initiative (PIFI)

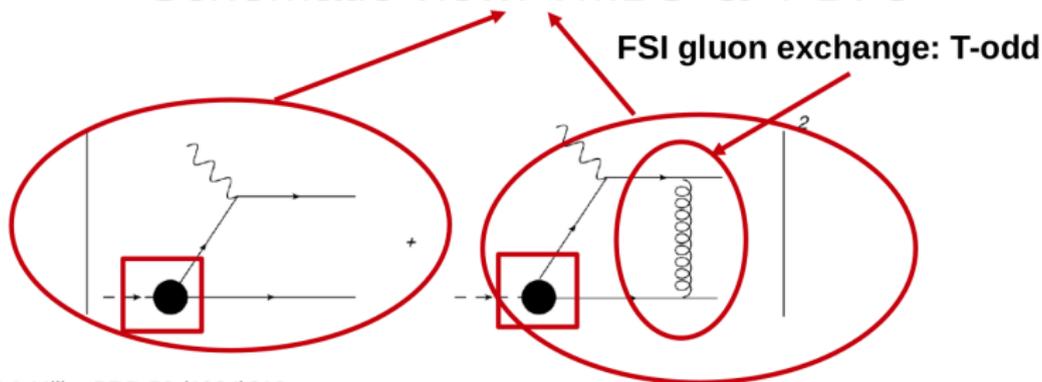


Special thanks to the hospitality of Institute of Modern Physics/CAS and Department of Modern Physics, University of Science and Technology of China.

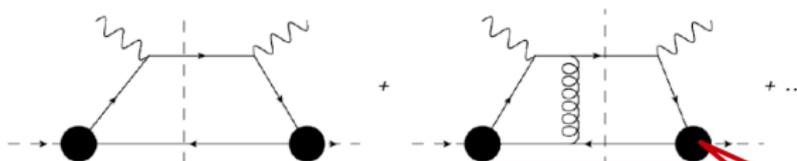
**THANK YOU!**

# BACKUP SLIDES

## Schematic view: TMDs &amp; PDFs



TF &amp; Miller PRD 50 (1994)210



$$q^2 = q^+ q^- - q_T^2$$

$$q^+ = q^0 + q^3 \quad q^- = q^0 - q^3$$

$q^- \rightarrow \text{infty}$  (DIS)

Bethe-Salpeter  
Amplitude @  $x^+=0$

DIS

## Two-body amplitude for the contact interaction

Ydrefors, Nogueira, Karmanov, and TF, PRD101, 096018 (2020)

(i) If  $-\infty < M_{12}^2 \leq 0$  ( $1 \geq y \geq 0$ ), then:

$$\mathcal{F}(M_{12}^2) = \left[ \frac{1}{16\pi^2 y} \log \frac{1+y}{1-y} - \frac{\arctan y'_{M_2}}{8\pi^2 y'_{M_2}} \right]^{-1}.$$

(ii) If  $0 \leq M_{12}^2 \leq 4m^2$  ( $0 \leq y' < \infty$ ), then:

$$\mathcal{F}(M_{12}^2) = \left[ \frac{\arctan y'}{8\pi^2 y'} - \frac{\arctan y'_{M_2}}{8\pi^2 y'_{M_2}} \right]^{-1}.$$

(iii) If  $4m^2 \leq M_{12}^2 < \infty$  ( $0 \leq y'' \leq 1$ ), then:

$$\mathcal{F}(M_{12}^2) = \left[ \frac{y''}{16\pi^2} \log \frac{1+y''}{1-y''} - \frac{\arctan y'_{M_2}}{8\pi^2 y'_{M_2}} - i \frac{y''}{16\pi} \right]^{-1}.$$

Here  $y'_{M_2} = \frac{M_2}{\sqrt{4m^2 - M_2^2}}$  and

$$y = \frac{\sqrt{-M_{12}^2}}{\sqrt{4m^2 - M_{12}^2}}, \quad y' = \frac{M_{12}}{\sqrt{4m^2 - M_{12}^2}},$$

$$y'' = \frac{\sqrt{M_{12}^2 - 4m^2}}{M_{12}}.$$

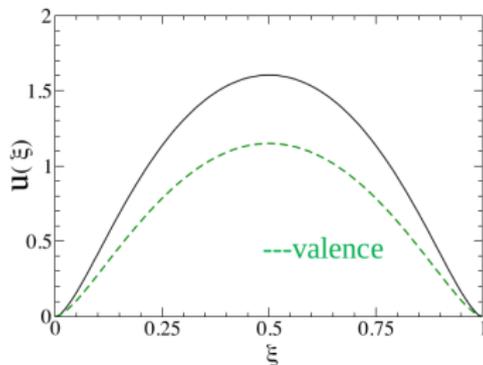
## The parton distribution function in a pion with Minkowskian dynamics

de Paula, Ydrefors, Alvarenga Nogueira, TF, Salmè, PRD 105, L071505 (2022)

T-even leading-twist (twist 2) - TMD (transverse-moment dependent) functions

$$f_1(\gamma, \xi) = \frac{N_c}{4} \int d\phi_{\mathbf{k}_\perp} \int \frac{dz^- dz_\perp}{2(2\pi)^3} e^{i[\xi P^+ z^- / 2 - \mathbf{k}_\perp \cdot \mathbf{z}_\perp]} \langle P | \bar{\psi}_q(-\frac{1}{2}z) \gamma^+ \psi_q(\frac{1}{2}z) | P \rangle \Big|_{z^+=0} \quad \text{LC gauge}$$

$$f_1(\gamma, \xi) = \frac{1}{(2\pi)^4} \frac{1}{8} \int_{-\infty}^{\infty} dk^+ \delta(k^+ + P^+/2 - \xi P^+) \int_{-\infty}^{\infty} dk^- \int_0^{2\pi} d\phi_{\mathbf{k}_\perp} \\ \times \left\{ \text{Tr} \left[ S^{-1}(k - P/2) \bar{\Phi}(k, P) \frac{\gamma^+}{2} \Phi(k, P) \right] - \text{Tr} \left[ S^{-1}(k + P/2) \Phi(k, P) \frac{\gamma^+}{2} \bar{\Phi}(k, P) \right] \right\}$$



$$u(\xi) = \int_0^{\infty} d\gamma f_1(\gamma, \xi)$$

$$u_{val}(\xi) = \int_0^{\infty} \frac{d\gamma}{(4\pi)^2} \left[ |\psi_{\uparrow\downarrow}(\gamma, z)|^2 + |\psi_{\uparrow\uparrow}(\gamma, z)|^2 \right]$$

- Full PDF normalization is 1
  - Valence PDF normalization is 0.7
- For  $\xi \rightarrow 1$ , PDF  $\sim (1 - \xi)^{\eta_0}$       $\eta_0 = 1.4$

## Parton distribution function

WP, Ydrefors, Nogueira, Frederico and Salme PRD 105 L071505 (2022).

Low order Mellin moments at scales  $Q = 2.0$  GeV and  $Q = 5.2$  GeV.

	BSE <sub>2</sub>	LQCD <sub>2</sub>	BSE <sub>5</sub>	LQCD <sub>5</sub>
$\langle x \rangle$	0.259	$0.261 \pm 0.007$	0.221	$0.229 \pm 0.008$
$\langle x^2 \rangle$	0.105	$0.110 \pm 0.014$	0.082	$0.087 \pm 0.009$
$\langle x^3 \rangle$	0.052	$0.024 \pm 0.018$	0.039	$0.042 \pm 0.010$
$\langle x^4 \rangle$	0.029		0.021	$0.023 \pm 0.009$
$\langle x^5 \rangle$	0.018		0.012	$0.014 \pm 0.007$
$\langle x^6 \rangle$	0.012		0.008	$0.009 \pm 0.005$

LQCD,  $Q = 2.0$  GeV:  $\langle x \rangle$  - Alexandrou et al PRD 103, 014508 (2021)

$\langle x^2 \rangle$  and  $\langle x^3 \rangle$  - Alexandrou et al PRD 104, 054504 (2021)

LQCD,  $Q = 5.0$  GeV:  $\langle x \rangle$  - Alexandrou et al PRD 103, 014508 (2021)

Hadronic scale and effective charge for DGLAP

$Q_0 = 0.330 \pm 0.030$  GeV - Cui et al EPJC 2020 80 1064

Within the error, we choose  $Q_0 = 0.360$  GeV to fit the first Mellin moment.

We used lowest order DGLAP equations for evolution

## Fermion Schwinger-Dyson equation

- Parameters:  $\alpha = \frac{g^2}{4\pi}$ ,  $\Lambda$ ,  $m_g$ ,  $\overline{m}_0$ .

- Spectral densities are obtained from the IR of the self-energy:

$$\rho_A(\gamma) = -\frac{1}{\pi} \text{Im} [A(\gamma)]$$

$$\rho_B(\gamma) = -\frac{1}{\pi} \text{Im} [B(\gamma)]$$

- Solutions of DSE obtained writing the trivial relation  $S_f^{-1}S_f = 1$  in a suitable form:

$$\frac{R}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{\gamma - s + i\epsilon} = \frac{A(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$

$$\frac{R\overline{m}_0}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{\gamma - s + i\epsilon} = \frac{B(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$

# Fermion Schwinger-Dyson equation

$$\begin{aligned}\rho_A(\gamma) &= R K_{0A}^\xi(\gamma, \bar{m}_0^2, m_g^2) \\ &+ \int_0^\infty ds \mathcal{K}_A^\xi(\gamma, s, m_g^2) \rho_v(s) - [m_g \rightarrow \Lambda] \\ \rho_B(\gamma) &= R \bar{m}_0 K_{0B}^\xi(\gamma, \bar{m}_0^2, m_g^2) \\ &+ \int_0^\infty ds \mathcal{K}_B^\xi(\gamma, s, m_g^2) \rho_s(s) - [m_g \rightarrow \Lambda]\end{aligned}$$

- **Driving term:**

$$K_{0A(0B)}^\xi = K_{A(B)} + m_g^{-2} \bar{K}_{A(B)}^\xi$$

- **Kernel:**

$$\begin{aligned}\mathcal{K}_A^\xi(\gamma, s, m_g^2) &= K_A(\gamma, s, m_g^2) \Theta(s - (\bar{m}_0 + m_g)^2) \\ &+ m_g^{-2} \bar{K}_A^\xi(\gamma, s, m_g^2) \Theta(s - (\bar{m}_0 + \sqrt{\xi} m_g)^2)\end{aligned}$$

## Connection Formulas

$$\begin{aligned}f_A(\gamma) &= 1 + \int_0^\infty ds \frac{\rho_A(s)}{\gamma - s} \\ f_B(\gamma) &= m_B + \int_0^\infty ds \frac{\rho_B(s)}{\gamma - s} \\ d(\gamma) &= \left[ \gamma f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma) \right]^2 \\ &+ 4\pi^2 \left[ \gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma) \right]^2\end{aligned}$$

$$\begin{aligned}\rho_v(\gamma) &= -2 \frac{f_A(\gamma)}{d(\gamma)} \left[ \gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma) \right] \\ &+ \frac{\rho_A(\gamma)}{d(\gamma)} \left[ \gamma f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma) \right] \\ \rho_s(\gamma) &= -2 \frac{f_B(\gamma)}{d(\gamma)} \left[ \gamma \rho_A(\gamma) f_A(\gamma) - \rho_B(\gamma) f_B(\gamma) \right] \\ &+ \frac{\rho_B(\gamma)}{d(\gamma)} \left[ \gamma f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma) \right]\end{aligned}$$

## $0^-$ Bound State with Running quark mass function

Castro, de Paula,TF,Salme PLB845 (2023) 138159

Integral Representation:  $S^V(p^2) = \int_0^\infty ds \frac{\rho^V(s)}{p^2 - s + i\epsilon}$  ;  $S^S(p^2) = \int_0^\infty ds \frac{\rho^S(s)}{p^2 - s + i\epsilon}$

Using the Nakanishi integral representation for  $\phi_i(k, p)$ , performing the loop integral and projecting onto the LF, one obtains the BSE as

$$\int_0^\infty d\gamma' \frac{g_i(\gamma', z)}{\left[\gamma + z^2 M^2/4 + \gamma' + \kappa^2 - i\epsilon\right]^2} = \frac{\alpha}{2\pi}$$

$$\times \sum_j \int_{-1}^1 dz' \int_0^\infty d\gamma' \mathcal{L}_{ij}(\gamma, z; \gamma', z') g_j(\gamma', z').$$

## PION DYNAMICS & QUARK SELF ENERGY

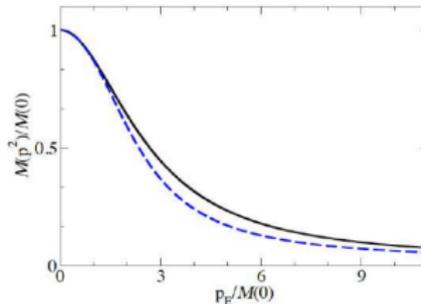
### $0^-$ Bound State with Running quark mass function

Castro, de Paula,TF,Salme PLB845 (2023) 138159

Dressed quark propagator:  $S(p) = i Z(p^2) \frac{\not{p} + \mathcal{M}(p^2)}{p^2 - \mathcal{M}^2(p^2)}$

Phenomenological model to reproduce Lattice Data for  $\mathcal{M}(p^2)$  (Mello, de Melo, Frederico, PLB 766, 86 (2017)):

$$\mathcal{M}(p^2) = m_0 - \frac{m^3}{p^2 - \lambda^2 + i\epsilon}$$



From Lattice simulations: bare quark mass  $m_0 = 8$  MeV and  $\mathcal{M}(0) = 0.344$  GeV.  
Our fit (solid line):  $m = 0.648$  GeV and  $\lambda = 0.9$  GeV.

Lattice data (dashed line) from: Oliveira, Silva, Skullerud and Sternbe, PRD 99 (2019) 094506

## $0^-$ Bound State with Running quark mass function

Castro, de Paula,TF,Salme

$$S^V(p^2) = \int_0^\infty ds \frac{\rho^V(s)}{p^2 - s + i\epsilon} \quad ; \quad S^S(p^2) = \int_0^\infty ds \frac{\rho^S(s)}{p^2 - s + i\epsilon}$$

Phenomenological model:  $\mathcal{M}(p^2) = m_0 - \frac{m^3}{p^2 - \lambda^2 + i\epsilon}$

$$\rho^{S(V)}(s) = \sum_{a=1}^3 R_a^{S(V)} \delta(s - m_a^2),$$

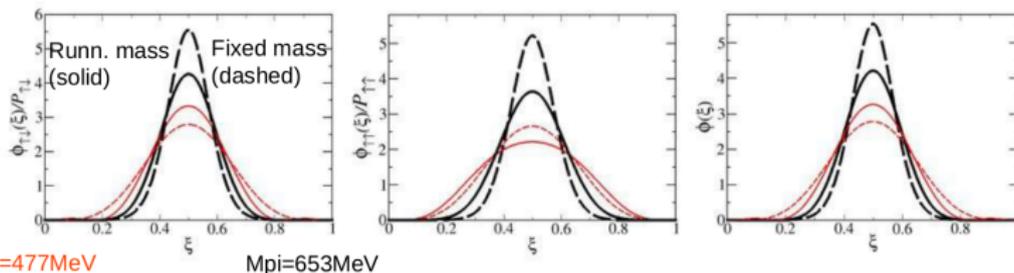
where  $R_a^{S(V)}$  are the **residues**, that read

$$R_a^V = \frac{(\lambda^2 - m_a^2)^2}{(m_a^2 - m_b^2)(m_a^2 - m_c^2)},$$

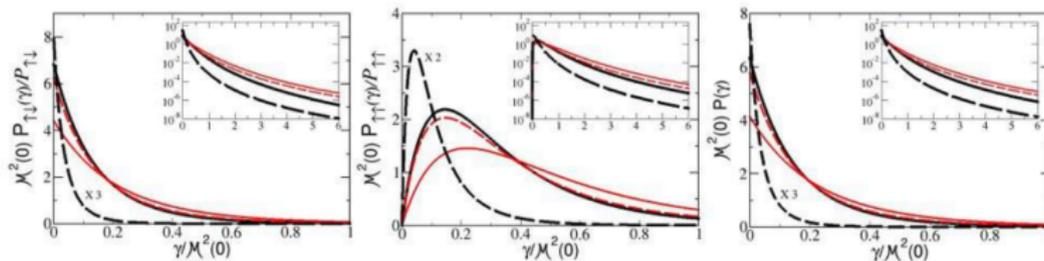
$$R_a^S = R_a^V \mathcal{M}(m_a^2),$$

with the indices  $\{a, b, c\}$  following the cyclic permutation  $\{1, 2, 3\}$ .

$\mathcal{M}(0)$	$i$	$m_i$	$R_i^V$	$R_i^S$
[GeV]		[GeV]		[GeV]
	1	0.4696	3.7784	1.7743
0.344	2	0.5733	-2.8863	-1.6546
	3	1.0349	0.1079	-0.1116



**Mpi=477MeV** **Mpi=653MeV**  
**Fig. 2.** Longitudinal momentum distributions defined in Eqs. (20) and (22), with  $\Lambda = 0.1\text{ GeV}$  and  $\mu = 0.469\text{ GeV}$ . Thick solid line: running mass model for  $M = 0.653\text{ GeV}$ . Thin solid line: the same as the thick one, but for  $M = 0.447\text{ GeV}$ . Thick dashed Line: fixed quark mass equal to  $0.344\text{ GeV}$  and  $M = 0.653\text{ GeV}$ . Thin dashed line: the same as the thick one, but for  $M = 0.447\text{ GeV}$ . Dotted line: the same as the thick one, but for  $M = m_{\pi} = 0.141\text{ GeV}$ .



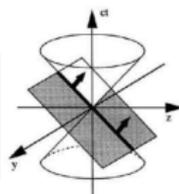
**Fig. 3.** Transverse momentum distributions defined in Eqs. (21) and (23), with  $\Lambda = 0.1\text{ GeV}$  and  $\mu = 0.469\text{ GeV}$ . The legend of the lines is the same as in Fig. 2.

## Generalized Stieltjes transform and the LF valence wave function

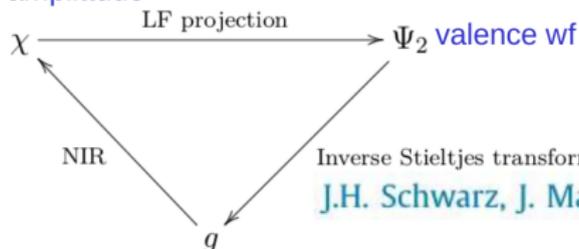
Carbonell, TF, Karmanov PLB769 (2017) 418 (bosons)

$$\Psi_i(\gamma, z; \kappa^2) = \int_0^\infty d\gamma' \frac{g_i(\gamma', z; \kappa^2)}{[\gamma + \gamma' + m^2 z^2 + (1 - z^2)\kappa^2]^2}$$

$$\gamma = k_\perp^2 \quad z = 2x - 1$$



BS amplitude



J.H. Schwarz, J. Math. Phys. 46 (2005) 014501,

**UNIQUENESS OF THE NAKANISHI REPRESENTATION**

**PHENOMENOLOGICAL APPLICATIONS from the valence wf  $\rightarrow$  BSA!**

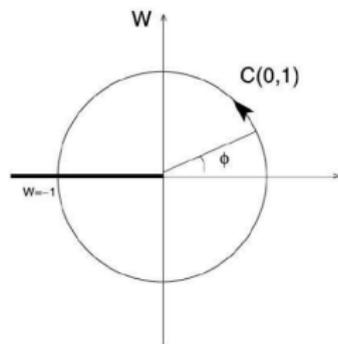
## Generalized Stieltjes transform and the LF valence wave function II

Carbonell, TF, Karmanov PLB769 (2017) 418

$$f(\gamma) \equiv \int_0^{\infty} d\gamma' L(\gamma, \gamma') g(\gamma') = \int_0^{\infty} d\gamma' \frac{g(\gamma')}{(\gamma' + \gamma + b)^2}$$

denoted symbolically as  $f = \hat{L} g$ .

$$g(\gamma) = \hat{L}^{-1} f = \frac{\gamma}{2\pi} \int_{-\pi}^{\pi} d\phi e^{i\phi} f(\gamma e^{i\phi} - b).$$



J.H. Schwarz, J. Math. Phys. 46 (2005) 014501,