Mechanical Properties of a Composite Spin-$\frac{1}{2}$ state Using Light-Cone Wave Functions

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Introduction

- The proton and neutron are the fundamental constituents of atomic nuclei.
  [E. Rutherford (1919). *Phil. Mag. Ser. 6*, 37:581–587] [J. Chadwick (1932). *Nature*, 129:312]

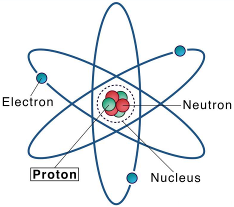


Figure: Atomic nuclei. Image Source: Google.

- Proton and neutron are the lightest strongly interacting spin-$\frac{1}{2}$ fermions, called the *baryons*.
- Another class bosons, called *mesons*, includes pions as the lightest.
  [C M G. Lattes, H. Muirhead, G P S. Occhialini, and C F. Powell (1947). *Nature*, 159:694–697] [E. Gardner and C M G. Lattes (1948). *Science*, 107:270–271]
- *baryons* and *mesons*, are collectively known as *hadrons*.

Finite size of proton

- Protons and neutrons are collectively known as *nucleons* based on the approximate isospin symmetry.
  [W. Heisenberg (1932). *Z. Phys.*, 77:1–11]
- If nucleons were a spin$-\frac{1}{2}$ *point particle* then it would have a magnetic moment of $\mu_N = \frac{e\hbar}{2M_N}$, according to Dirac equation.
  - Proton and neutron magnetic moment was measured to be $\mu_p = 2.5\ \mu_N$ and $\mu_n = -1.5\mu_N$.
    [R. Frisch and O. Stern (1933). *Zeitschrift für Physik*, 85(1):4–16] [L W. Alvarez and F. Bloch (1940). *Phys. Rev.*, 57:111–122]

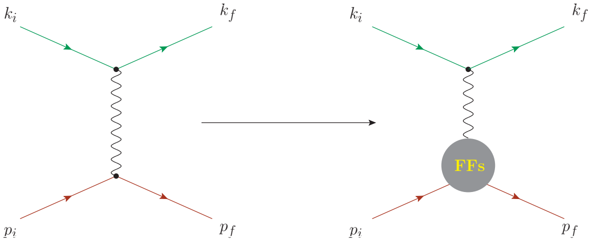


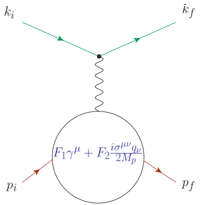
Figure: Electron - proton elastic scattering.

- The form factors (FFs) are defined through the matrix elements of the electromagnetic current operator $\langle p', \boldsymbol{s}' | J_p^\mu | p, \boldsymbol{s} \rangle$ and functions of $q^2 = (k_i - k_f)^2$.

- The most general form of four-current constructed for a spin-$\frac{1}{2}$ has two form factors

$$J_p^\mu(x) = \overline{u}^{(s')}(p_f)[F_1(q^2)\gamma^\mu + \frac{\kappa}{2M_p} F_2(q^2)\, i\sigma^{\mu\nu} q_\nu] u^{(s)}(p_i)\, e^{i(p'-p)\cdot x},$$

where $F_1(q^2)$ and $F_2(q^2)$ are two independent form factors, κ is the anomalous magnetic moment, M_p is mass of the proton and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.



- For $q^2 \to 0$ the target is effectively a particle of charge e and magnetic moment $(1+\kappa)e/2M_p$, where κ is the amomalous magnetic moment. Thus in this limit form factors for the proton are

$$F_1(0) = 1, \quad F_2(0) = 1.$$

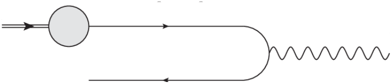
Correcponding values for neutron are

$$F_1(0) = 0, \quad F_2(0) = 1.$$

Charge and magnetic moment distribution

- $G_E(q^2) \equiv F_1(q^2) + \frac{\kappa q^2}{4M_p^2} F_2(q^2)$, $G_M(q^2) \equiv F_1(q^2) + \kappa F_2(q^2)$.

- Breit frame $p_f^\mu = (E, -\boldsymbol{p})$ and $p_i^\mu = (E, \boldsymbol{p})$, there is no energy transfer to the proton.



- In this frame, we have

$$\rho = 2M_p e\, G_E, \quad \boldsymbol{J} = e\overline{u}(-\boldsymbol{p})\boldsymbol{\gamma}\, u(\boldsymbol{p})\, G_M\,.$$

- The condition is

$$q^2 \left(\frac{q^2}{4M_p^2} - 1 \right) = \boldsymbol{q}^2.$$

So in the limit $q^2 << 4M_p^2$ the timelike component of q^2 is relatively small and $q^2 \approx -\boldsymbol{q}^2$.

- The nature of the plots can be parametrized by the dipole formula

$$G_M(Q^2) = 2.79 G_E(Q^2) \approx 2.79 \frac{1}{\left(1 + \frac{Q^2}{0.71\ GeV^2}\right)^2}.$$

- This implies that the proton charge distribution has an exponential form in configuration space

$$\rho(\boldsymbol{r}) \approx e^{-mr},$$

with $m^2 = 0.71 GeV^2$.

- The mean square charge radius of the proton is

$$\langle r^2 \rangle = 6 \left(\frac{d\, G_E(Q^2)}{dQ^2} \right)_{Q^2=0} = \left(0.81 \times 10^{-13}\ \text{cm}\right)^2.$$

[R W. Mcallister and R. Hofstadter (1956). *Phys. Rev.*, 102:851–856]

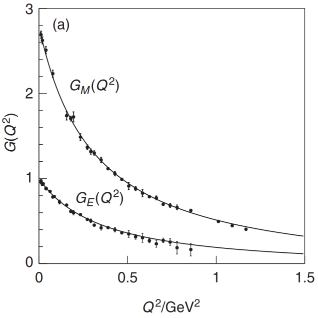
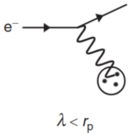


Figure: Proton elastic form factors as a function of $Q^2 = -q^2$, []. Image source: [M. Thomson (2013). Cambridge University Press, New York]

Electron-proton deep inelastic scattering (DIS)



- If we want a more detailed structure, we have to decrease the wavelength of the virtual photon for better resolution, which can be done by increasing the energy loss of the scattered electron.

  (Image source: **[M. Thomson (2013). Cambridge University Press, New York]**)

- If we increase the Q^2 of the virtual photon then elastic scattering cross-section becomes

$$\left.\frac{d\sigma}{d\Omega}\right|_{(\text{elastic})} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau \tan^2 \frac{\theta}{2}\right) G_M^2 , \quad \text{where} \left(\frac{d\sigma}{d\Omega}\right)_0 = \left(\frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}}\right) \frac{E'}{E} \cos^2 \frac{\theta}{2} .$$

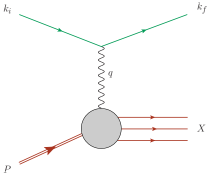
- At high Q^2 limit $G_M \propto Q^{-4}$, so

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{elastic}} \propto \frac{1}{Q^6} \left(\frac{d\sigma}{d\Omega}\right)_0 ,$$

  with increasing virtuality of the photon, due to the finite size of the proton, the probability of elastic scattering decreases and inelastic scattering dominates.

Deep inelastic scattering

- In electron-proton deep inelastic scattering (DIS), a high-energy proton breaks apart into many particles, losing its identity.
  [J I. Friedman and H W. Kendall (1972). ***Annual Review of Nuclear Science*****, 22(1):203–254].**



- Invariant mass of the final state takes a range of values $W^2 = (P+q)^2 = M_p^2 + 2P\cdot q + q^2$ and now $(P\cdot q)$ is also an independent variable. We define

$$Q^2 = -q^2, \quad \nu = \frac{P\cdot q}{M_p}.$$

- The structure functions $F_{1,2}$ that describe the response of nucleons in DIS are functions of Q^2 and ν.

Bjorken scaling

- The *Bjorken limit* is when $Q^2, \nu \to \infty$ with $x = \frac{Q^2}{2M_p\nu}$ fixed, leading to structure functions $F_i(x, Q^2) \to F_i(x)$. This indicates that the virtual photon scatters off point-like particles within the proton.
- This was predicted based on current algebra and dispersion relation techniques and was observed in DIS.

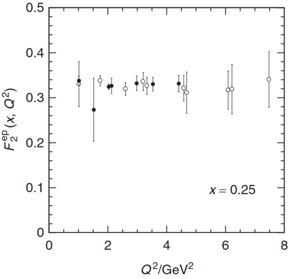
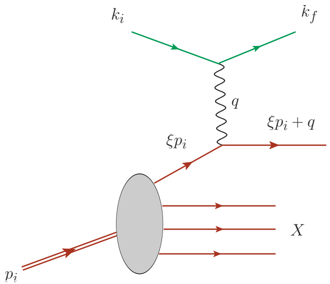


Figure: Measuremt of $F_2^{ep}(x, Q^2)$ showing Bjorken scaling at SLAC, [J D. Bjorken (1969). *Phys. Rev.*, 179:1547–1553], [E D. Bloom et al. (1969). *Phys. Rev. Lett.*, 23:930–934]. Image source: [M. Thomson (2013). Cambridge University Press, New York]

Parton model

- The nucleon structure is described by parton distribution functions (PDFs), $f_1^a(\xi)$, which represent the probability of finding a parton of type "a" with momentum fraction in the interval $[\xi, \xi + d\xi]$.
  [R P. Feynman (1969). *Phys. Rev. Lett.*, 23:1415–1417]



- In this model, the electron scatters off nearly *free* electrically charged point-like spin-$\frac{1}{2}$ particles called partons.
- These charged partons are also termed 'quarks' due to the theoretical work of Gell-Mann.
  [M. Gell-Mann (1964). *Phys. Lett.*, 8:214–215]

Scaling violation

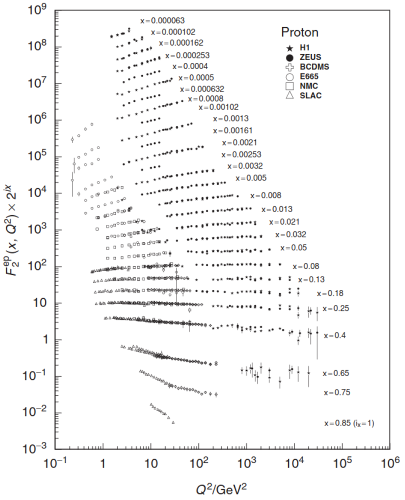


Figure: Deviation of scaling of the structure-function $F_2^{ep}(x, Q^2)$. Image source [M. Thomson (2013). Cambridge University Press, New York].

Mechanical properties of hadron

- The QCD Lagrangian

$$\mathcal{L} = \sum_q \overline{\psi}_q \left(i\gamma^\mu D_\mu + m_q \right) \psi_q - \frac{1}{4} F^c_{\mu\nu} F^{c\mu\nu},$$

where $\psi_q \left(\overline{\psi}_q\right)$: quark (anti-quark) field, $iD_\mu = i\partial_\mu + gA^c_\mu T^c$, T^c: $SU(N_c)$ generators, A^c: gauge fields, $F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gf^{cde} A^d_\mu A^e_\nu$, f^{cde}: structure constants of $SU(N_c)$ group.

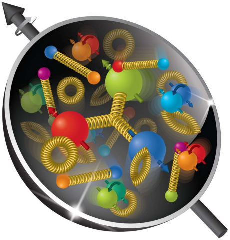
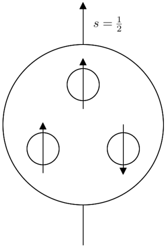


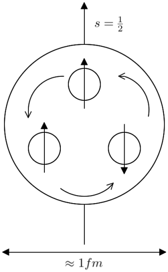
Image source: BNL photo albums

- What is the source of nucleon mass?
- How to understand the nucleon spin in terms of its fundamental degrees of freedom?



- Polarized deep inelastic scattering (DIS) experiments suggest that only one-third of nucleon spin comes from the quark's intrinsic spin.
  [J. Ashman et al. (1988). *Phys. Lett. B*, 206:364]
- RHIC-spin experiments have provided important constraints on the contribution of gluon's helicity to the proton spin.
  [D. de Florian, R. Sassot, M. Stratmann, and W. Vogelsang (2014). *Phys. Rev. Lett.*, 113(1):012001]

- $\frac{1}{2}$ = Quark spin + Gluon helicity + *Orbital angular momentum* (?)



- Present experiments like Jlab 12 GeV and upcoming Electron-Ion Collider (EIC) at Brookhaven National Lab aim to measure the OAM and spin of all partons with increased accuracy.

[J. Dudek et al. (2012). ***Eur. Phys. J. A*****, 48:187] [R. Abdul Khalek et al. (2022).** ***Nucl. Phys. A*****, 1026:122447]**

- The matrix elements of the energy-momentum tensor (EMT) of QCD are essential for addressing these questions.

- Parameterizing the EMT for a spin-$\frac{1}{2}$ system can be achieved in terms of four gravitational form factors (GFFs) **[M V. Polyakov and P. Schweitzer (2018). *Int. J. Mod. Phys. A*, 33(26):1830025]**

$$\begin{aligned}&\langle p',\lambda'|T_i^{\mu\nu}(0)|p,\lambda\rangle\\ =&\overline{u}(p',\lambda')\Big[-B_i(\Delta^2)\frac{P^\mu P^\nu}{M}+\Big(A_i(\Delta^2)+B_i(\Delta^2)\Big)\frac{1}{2}\left(\gamma^\mu P^\nu+\gamma^\nu P^\mu\right)\\ &+C_i(\Delta^2)\frac{\Delta^\mu\Delta^\nu-\Delta^2 g^{\mu\nu}}{M}+\overline{C}_i(\Delta^2)Mg^{\mu\nu}\Big]u(p,\lambda),\end{aligned}$$

where $i=q,\ G$, $P^\mu=\frac{1}{2}\left(p'+p\right)^\mu$ and $\Delta^\mu=\left(p'-p\right)^\mu$.

- Gordon decomposition:

$$\overline{u}(p',\lambda)\left[P^\mu i\sigma^{\nu\lambda}q_\lambda+P^\nu i\sigma^{\mu\lambda}q_\lambda\right]u(p,\lambda)=\overline{u}(p',\lambda)\left[2M\left(\gamma^\mu P^\nu+\gamma^\nu P^\mu\right)-4P^\mu P^\nu\right]u(p,\lambda).$$

- Conservation of total energy-momentum tensor (EMT) rules out terms like $(P^\mu q^\nu+q^\mu P^\nu)$ and $(\gamma^\mu q^\nu+\gamma^\nu q^\mu)$ also puts a constraint $\sum_i\overline{C}_i(q^2)=0$.

Constraints on the gravitational form factors

- The conservation of momentum puts a constraint:

$$\sum_{q,G} A_i(0) = 1.$$

- Ji's sum rule

$$J_{q,G} = \frac{1}{2}\left(A_{q,G}(0) + B_{q,G}(0)\right),$$
$$\sum_{q,G} B_i(0) = 0.$$

  where $J_{q,G}$: Total angular momentum of quarks and gluons.

- Conservation of the total EMT puts a constraint:

$$\sum_{q,G} \overline{C}_i(\Delta^2) = 0.$$

- The GFF $C(\Delta^2)$ also known as the *D*-term $(D(\Delta^2) = 4C(\Delta^2))$, is unconstrained at zero momentum transfer.
- *D*-term or *Druck*-term, which provides us with information about the internal forces within the system.

Observables for gravitational form factors

- Graviton-proton scattering, as the EMT couples to the graviton.
  [H. Pagels (1966). *Phys. Rev.*, 144:1250–1260]

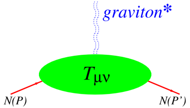
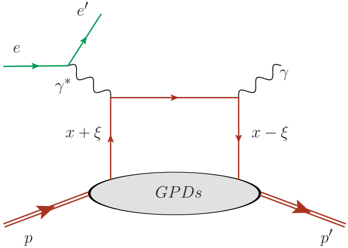


Image source: [M V. Polyakov and P. Schweitzer (2018). *Int. J. Mod. Phys. A*, 33(26):1830025]

- Hard exclusive scatterings e.g. deeply virtual Compton scattering (DVCS): $ep \to e'p'\gamma$



- The generalized parton distribution (GPD) functions

$$\Phi^{[\Gamma]}(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p'|\overline{\psi}\left(-\frac{1}{2}z\right)\Gamma\psi\left(\frac{1}{2}z\right)|p\rangle\Big|_{z^+=0.\boldsymbol{z}^\perp=0},$$

Γ : Dirac matrix, ψ : Quark field.

- GPDs depend on $x,\ t,\ \xi$

$$x = \frac{k^+ + k'^+}{p^+ + p'^+}, \qquad t = \Delta^2, \qquad \xi = \frac{p^+ - p'^+}{p^+ + p'^+},$$

(Average longitudinal momentum fraction of a parton) (Four momentum squared) (Skewness parameter)

define

$$P = \frac{p+p'}{2}, \quad \Delta = p' - p.$$

- For $\Gamma = \gamma^+$

$$\Phi^{[\gamma^+]}(x,\xi,t) = \frac{1}{2P^+}\left[H^q(x,\xi,t)\overline{u}(p')\gamma^+u(p) + E^q(x,\xi,t)\overline{u}(p')\frac{i\sigma^{+\alpha}}{2m}u(p)\right].$$

Gravitational form factors from GPDs

- The use of GPDs to address the physical content of GFFs was done by Ji in the context of the angular momentum decomposition of nucleons.

$$\int_{-1}^{1} dx\, x \left(H^a(x,\xi,t) + E^a(x,\xi,t)\right) = A^a(t) + B^a(t),$$

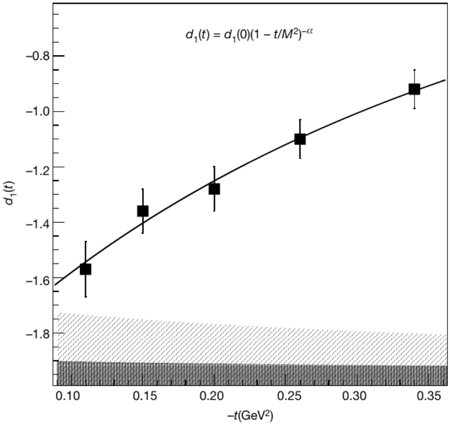
where $a = g,\ u,\ d,\ \ldots$ are type of partons. **[X D. Ji (1997).** ***Phys. Rev. Lett.*****, 78:610–613]**

- The GPDs $H^q(x,\xi,t)$ and $E^q(x,\xi,t)$ give access to the quark GFFs as follows

$$\int_{-1}^{1} dx\, xH^q(x,\xi,t) = A^q(t) + \xi^2 D^q(t),$$
$$\int_{-1}^{1} dx\, xE^q(x,\xi,t) = B^q(t) - \xi^2 D^q(t).$$

[M V. Polyakov and P. Schweitzer (2018). ***Int. J. Mod. Phys. A*****, 33(26):1830025]**

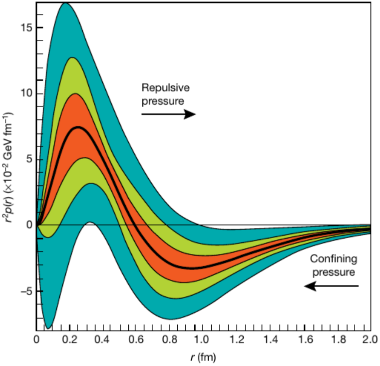
- The D-term has been determined from experimental data on deeply virtual Compton scattering (DVCS) at Jefferson Lab. [V D. Burkert, L. Elouadrhiri, and F X. Girod (2018). *Nature*, 557(7705):396–399]

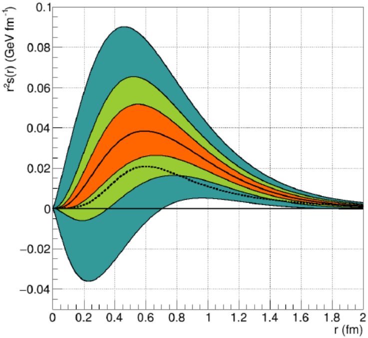


- The pressure and shear force experienced by quarks.

  [V D. Burkert, L. Elouadrhiri, and F X. Girod (2018). *Nature*, 557(7705):396–399]

  [Burkert, V. D., Elouadrhiri, L., and Girod, F. X. (2021). arXiv:2104.02031v2 [nucl-ex]]





Light-front coordinates:

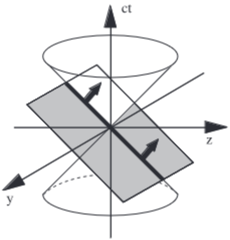
$$x^{\pm} = \left(x^0 \pm x^3\right), \quad x^{\mu} = \left(x^+, x^-, \boldsymbol{x}^{\perp}\right)$$

where x^+ is the light-front time, x^- and $\boldsymbol{x}^{\perp}$ are the longitudinal and transverse spatial coordinates respectively.

Light-front four-momentum:

$$p^{\mu} = \left(p^+, p^-, \boldsymbol{p}^{\perp}\right),$$

where p^+ is the longitudinal momentum, p^- and $\boldsymbol{p}^{\perp}$ are the energy and transverse momentum respectively.



- Metric tensor as

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Also we define the dot product of x^{μ} and p^{μ} in the following way

$$x \cdot p = \frac{1}{2}x^+p^- + \frac{1}{2}x^-p^+ - \boldsymbol{x}^{\perp} \cdot \boldsymbol{p}^{\perp}.$$

Image source: [S J. Brodsky, H C. Pauli, and S S. Pinsky (1998). *Phys. Rept.*, 301:299–486]

Benefits of using light front coordinates

- Simple dispersion relation: $p^- = \frac{p^{\perp 2}+m^2}{p^+}$.
- In light-front coordinates then we can define the Poincaré generators in the following way

$$P^\mu = \frac{1}{2}\int dx^- d^2\boldsymbol{x}^\perp \; T^{+\mu}$$
$$M^{\mu\nu} = \frac{1}{2}\int dx^- d^2\boldsymbol{x}^\perp \; \left[x^\mu T^{+\nu} - x^\nu T^{+\mu}\right].$$

- Dirac termed the interaction-dependent components as *dynamic* and interaction-independent components as *kinematic* components. The kinematic generators keep the $x^+ = 0$ surface invariant.
  [P A M. Dirac (1949). *Rev. Mod. Phys.*, 21:392–399]
- The boost operators: $M^{+-} = 2K^3, \;\; M^{+i} = E^i, \;\; (i = 1, 2)$, the rotational operators are $M^{12} = 2J^3, \;\; M^{-i} = F^i, \;\; (i = 1, 2)$, P^- is the light-front energy and $\boldsymbol{P} = \left(P^+, P^\perp\right)$.

Kinematic	Dynamic
$P^+, P^\perp, K^3, E^i, J^3$	P^-, F^i

- The longitudinal boost operator K^3 acts like a scale transformation and transverse boosts are Galilean in nature.
  - Consider a boost along the negative 3-axis

$$\tilde{x}^0 = x^0 \cosh\phi + x^3 \sinh\phi, \;\; \tilde{x}^3 = x^0 \sinh\phi + x^3 \cosh\phi,$$

    under this transformation we can prove

$$\tilde{x}^+ = \tilde{x}^0 + \tilde{x}^3 = e^{\phi} x^+, \;\; \tilde{x}^- = \tilde{x}^0 - \tilde{x}^3 = e^{-\phi} x^+.$$

  - Transverse boost generators

$$E^1 = -i \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; E^2 = -i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  - For infinitesimal trasformation $e^{i\phi E^j} = 1 + i\phi E^j$, we can prove $\tilde{x}^+ = \tilde{x}^0 + \tilde{x}^3 = x^+$, where $(j = 1, 2)$.

Internal Forces of Hadron

- For a local operator $\hat{O}(x)$ the corresponding spatial density is given by the expectation value $\langle\Psi|\hat{O}(x)|\Psi\rangle$, where $|\Psi\rangle$ represents a physically realizable hadron state.

$$\langle\hat{O}(\boldsymbol{x}^{\perp})\rangle = \frac{1}{2}\int dx^{-}\langle\Psi|\hat{O}(x^{+}=0,x^{-},\boldsymbol{x}^{\perp})|\Psi\rangle$$
$$= \int \frac{d^2\boldsymbol{\Delta}^{\perp}}{(2\pi)^2}\frac{\langle P^{+},\boldsymbol{p}'^{\perp},\lambda|\hat{O}(0)|P^{+},\boldsymbol{p}^{\perp},\lambda\rangle}{2P^{+}}e^{-i\boldsymbol{\Delta}^{\perp}\cdot\boldsymbol{x}^{\perp}}e^{-\frac{\sigma^2}{2}\Delta^{\perp 2}}.$$

The wave packet has a specific longitudinal momentum P^{+} and helicity λ:

$$\langle P^{+},\boldsymbol{p}^{\perp},s|\Psi\rangle = \sqrt{2\pi}(2\sigma)e^{-\sigma^2\boldsymbol{p}^{\perp 2}}\sqrt{2p^{+}(2\pi)\delta(p^{+}-P^{+})}\,\delta_{s\lambda},$$
$$P = \frac{p+p'}{2},\quad \Delta = p'-p.$$

- The two-dimensional distributions in light front coordinates are devoid of *relativistic corrections*.

- The transverse component of the EMT constitutes the stress tensor.

$$\langle T^{ij}(\boldsymbol{x}^\perp)\rangle = \frac{1}{2P^+}\int \frac{d^2\boldsymbol{\Delta}^\perp}{(2\pi)^2}\left[\frac{D_i(\Delta^2)}{2}\left(\Delta^i\Delta^j+\Delta^2\delta^{ij}\right)-\overline{C}_i(\Delta^2)2M^2\delta^{ij}\right]e^{-i\boldsymbol{\Delta}^\perp\cdot\boldsymbol{x}^\perp}e^{-\frac{\sigma^2}{2}\boldsymbol{\Delta}^{\perp 2}}.$$

- The stress tensor can be parametrized in two independent functions of $\boldsymbol{x}^\perp$ as follows

$$\langle T^{ij}(\boldsymbol{x}^\perp)\rangle = \delta^{ij}p(x^\perp)+\left(\frac{x^ix^j}{x^{\perp 2}}-\frac{1}{2}\delta^{ij}\right)s(x^\perp),$$

where $p(x^\perp)$ is the light font pressure, $s(x^\perp)$ is the shear stress and $x^\perp = |\boldsymbol{x}^\perp|$.

[A. Freese and G A. Miller (2021). *Phys. Rev. D*, 103:094023]

- The pressure

$$p(x^\perp) = \frac{1}{2}\delta_{ij}\langle T^{ij}(\boldsymbol{x}^\perp)\rangle$$
$$= \frac{1}{8P^+}\int \frac{d^2\boldsymbol{\Delta}^\perp}{(2\pi)^2}\left(-\Delta^{\perp 2}\right)D_i(\Delta^2)e^{-i\boldsymbol{\Delta}^\perp\cdot\boldsymbol{x}^\perp}e^{-\frac{\sigma^2}{2}\Delta^{\perp 2}}$$
$$-\frac{1}{P^+}\int \frac{d^2\boldsymbol{\Delta}^\perp}{(2\pi)^2}(M^2)\overline{C}_i(\Delta^2)e^{-i\boldsymbol{\Delta}^\perp\cdot\boldsymbol{x}^\perp}e^{-\frac{\sigma^2}{2}\Delta^{\perp 2}}.$$

- The von Laue stability condition

$$\int d^2\boldsymbol{x}^\perp p(x^\perp) = 0.$$

- The pressure and shear stress connected by the EMT conservation $\nabla_i\langle T^{ij}(\boldsymbol{x}^\perp)\rangle = 0$

$$p'(x^\perp)+\frac{1}{2}s'(x^\perp)+\frac{1}{x^\perp}s(x^\perp) = 0,$$

where $x^\perp = |\boldsymbol{x}^\perp|$ and $p'(x^\perp) = \frac{dp(x^\perp)}{dx^\perp}$.[A. Freese and G A. Miller (2021). *Phys. Rev. D*, 103:094023]

- The pressure and shear stress can be written in a differential form in impact parameter space

$$p(b^\perp) = \frac{1}{8m(b^\perp)}\frac{d}{db^\perp}\left[b^\perp \frac{d}{db^\perp}D_i(b^\perp)\right] - m\overline{C}_i(b^\perp),$$
$$s(b^\perp) = -\frac{b^\perp}{4m}\frac{d}{db^\perp}\left[\frac{1}{b^\perp}\frac{d}{db^\perp}D_i(b^\perp)\right],$$

where

$$F(b^\perp) = \frac{1}{(2\pi)^2}\int d^2\boldsymbol{\Delta}^\perp\, e^{-i\boldsymbol{\Delta}^\perp\cdot \boldsymbol{b}^\perp}\mathcal{F}(\Delta^2) = \frac{1}{2\pi}\int_0^\infty d\Delta^{\perp 2} J_0\left(\Delta^\perp b^\perp\right)\mathcal{F}(\Delta^2).$$

where $\mathcal{F} = \left(A_i, B_i, D_i, \overline{C}_i\right)$, $i \equiv (q, G)$. J_0 is Bessel's function of zeroth order, $\boldsymbol{b}^\perp$ is the impact parameter and m is the mass of the system. **[J. More, A. Mukherjee, S. Nair, and S. Saha (2022). *Phys. Rev. D*, 105(5):056017][J. More, A. Mukherjee, S. Nair, and S. Saha (2023). *Phys. Rev. D*, 107(11):116005]**

- Gaussian wave packet state with a gaussian width σ

$$\frac{1}{16\pi^3}\int \frac{d^2\boldsymbol{p}^\perp dp^+}{p^+} p^+\delta(p^+ - p_0^+)e^{-\frac{p^{\perp 2}}{2\sigma^2}}|p^+, \boldsymbol{p}^\perp, \lambda\rangle.$$

Stability condition

- The normal force density on a one-dimensional surface

$$\langle T^{ij}(\boldsymbol{x}^{\perp})\rangle \boldsymbol{x}_{\perp}^{i} \geq 0$$
$$p(x^{\perp}) + \frac{1}{2}s(x^{\perp}) \geq 0.$$

- Using pressure and shear force expressions

$$\int \frac{d^2\boldsymbol{\Delta}^{\perp}}{(2\pi)^2}\left[D(\Delta^2) + \Delta^2 \frac{dD(\Delta^2)}{d\Delta^2}\right] e^{-i\boldsymbol{\Delta}^{\perp}\cdot\boldsymbol{b}^{\perp}} \leq 0.$$

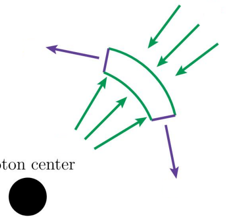
- Integrating the above expression over all space we get

$$D(0) \leq 0.$$

Normal and tangential force

- The normal and tangential forces:

$$F_n(b^\perp) = 2\pi b^\perp \left(p(b^\perp) + \frac{1}{2} s(b^\perp) \right),$$
$$F_t(b^\perp) = 2\pi b^\perp \left(p(b^\perp) - \frac{1}{2} s(b^\perp) \right).$$



The normal force must be $F_n(b^\perp) > 0$ and the tangential force $F_t(b^\perp)$ must change sign with the distance. [J. More, A. Mukherjee, S. Nair, and S. Saha (2022). *Phys. Rev. D*, 105(5):056017][J. More, A. Mukherjee, S. Nair, and S. Saha (2023). *Phys. Rev. D*, 107(11):116005]

Mechanical Properties of a Dressed Quark State

The dressed quark state

- We consider replacing the hadron state with a simpler relativistic spin-$\frac{1}{2}$ state of a quark dressed with a gluon at one loop in QCD as a perturbative model with a gluon degree of freedom.
  [A. Harindranath, R. Kundu, and W M. Zhang (1999). ***Phys. Rev. D*****, 59:094013]**
- The dressed quark state can be expanded in Fock space in terms of multiparton light-front wave functions (LFWFs) which can be calculated using the light-front Hamiltonian in perturbation theory.
  [A. Harindranath, A. Mukherjee, and R. Ratabole (2001). ***Phys. Rev. D*****, 63:045006]**
- We can write the LFWFs in a boost invariant way in terms of relative momenta that are frame-independent.
  [S J. Brodsky, H C. Pauli, and S S. Pinsky (1998). ***Phys. Rept.*****, 301:299–486]**

- Up to two-particle sector, dressed quark state is given by

$$
\begin{aligned}
|p,\lambda\rangle =& \psi_1(p,\lambda) b^\dagger_\lambda(p)|0\rangle \\
&+ \sum_{\lambda_1,\lambda_2} \int \frac{dk_1^+ d^2\boldsymbol{k}_1^\perp}{\sqrt{2(2\pi)^3 k_1^+}} \int \frac{dk_2^+ d^2\boldsymbol{k}_2^\perp}{\sqrt{2(2\pi)^3 k_2^+}} \psi_2\left(p,\lambda|k_1,\lambda_1;k_2,\lambda_2\right) \\
&\sqrt{2(2\pi)^3 P^+}\delta^3\left(p-k_1-k_2\right) b^\dagger_{\lambda_1}(k_1) a^\dagger_{\lambda_2}(k_2)|0\rangle,
\end{aligned}
$$

$\psi_1(p,\lambda)$ is normalization, $\psi_2\left(p,\lambda|k_1,\lambda_1;k_2,\lambda_2\right)$ is the probability amplitude of finding a quark and gluon with momentum (helicity) $k_1(\lambda_1)$ & $k_2(\lambda_2)$ respectively.

[A. Harindranath, R. Kundu, and W M. Zhang (1999). ***Phys. Rev. D*****, 59:094013]**

- $\psi_2\left(p,\lambda|k_1,\lambda_1;k_2,\lambda_2\right)$ can be calculated from the following equation

$$
H|p,\lambda\rangle = \frac{m^2 + \boldsymbol{p}^{\perp 2}}{p^+}|p,\lambda\rangle,
$$

where H is the light-front QCD Hamiltonian.

[W M. Zhang and A. Harindranath (1993). ***Phys. Rev. D*****, 48:4881–4902]**

- Boost invariant two-particle LFWF can be written as

$$\phi^{\lambda}_{\lambda_1,\lambda_2}(x_i,\boldsymbol{\kappa}_i) = \sqrt{p^+}\psi_2\left(p,\lambda|k_1,\lambda_1;k_2,\lambda_2\right)$$

$$\phi^{\lambda}_{\lambda_1,\lambda_2}(x_i,\boldsymbol{\kappa}_i)$$

$$= \frac{g}{\sqrt{2(2\pi)^3}}\left[\frac{x(1-x)}{\boldsymbol{\kappa}^{\perp 2}+m^2(1-x)^2}\right]\frac{T^a}{\sqrt{1-x}}\times$$

$$\chi^{\dagger}_{\lambda_1}\left[\frac{-2\left(\boldsymbol{\kappa}^{\perp}\cdot\epsilon^{\perp *}_{\lambda_2}\right)}{1-x} - \frac{1}{x}\left(\tilde{\sigma}^{\perp}\cdot\boldsymbol{\kappa}^{\perp}\right)\left(\tilde{\sigma}^{\perp}\cdot\epsilon^{\perp *}_{\lambda_2}\right) + im\left(\tilde{\sigma}^{\perp}\cdot\epsilon^{\perp *}_{\lambda_2}\right)\frac{1-x}{x}\right]\chi_{\lambda}\psi^{\lambda}_{1}.$$

Jacobi momentum

$$k_i^+ = x_i p^+, \qquad \boldsymbol{k}_i^{\perp} = \boldsymbol{\kappa}_i^{\perp} + x_i\boldsymbol{p}^{\perp},$$

$$x_1 + x_2 = 1, \qquad \boldsymbol{\kappa}_1^{\perp} + \boldsymbol{\kappa}_2^{\perp} = 0,$$

where m is quark mass, g is gluon coupling constant, T^a are $SU(3)$ colour matrices, $\epsilon^{\perp}_{\lambda_2}$ is gluon polarization vector and χ_λ is two-component spinor.

[A. Harindranath, R. Kundu, and W M. Zhang (1999). *Phys. Rev. D*, 59:094013]

Two-Component formalism

- In light-front coordinates, the unphysical degrees of freedom of the gauge field are eliminated by light-front gauge $A_a^+ = A_a^0 + A_a^3 = 0$.

- In the light-front Hamiltonian framework, the quark fields are decomposed as

$$\psi = \psi_+ + \psi_-,$$

where $\psi_\pm = \Lambda_\pm \psi$ and $\Lambda_\pm$ are the projection operators.

- The ψ_- component and longitudinal gauge field A_a^- are constrained fields and can be determined from the following equations

$$i\partial^+ \psi_- = \left(i\alpha^\perp \cdot \partial^\perp + g\alpha^\perp \cdot A^\perp + \beta m \right) \psi_+,$$

$$\frac{1}{2}\partial^+ E_a^- = \left(\partial^i E_a^i + g f^{abc} A_b^i E_c^i \right) - g\psi_+^\dagger T^a \psi_+,$$

where $\alpha^\perp = \gamma^0\gamma^\perp$, $\beta = \gamma^0$ and $E_a^{-,i} = -\frac{1}{2}\partial^+ A_a^{-,i}$, $(i = 1, 2)$.

[W M. Zhang and A. Harindranath (1993). *Phys. Rev. D*, 48:4881–4902]

- Independent dynamical degrees of freedom in light-front QCD are ψ_+ and the transverse gauge fields A^i_a.

- In light-front coordinates, a four-component fermion field can be reduced to a two-component field in a light-front representation of gamma matrices defined as

$$\gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \ \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}$$
$$\gamma^i = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}, \ \gamma^5 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}.$$

- In this representation, the projection operators become

$$\Lambda_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \Lambda_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- The quark fields decompose as

$$\psi_+ = \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \quad \psi_- = \begin{bmatrix} 0 \\ \eta \end{bmatrix}.$$

[W M. Zhang and A. Harindranath (1993). ***Phys. Rev. D*****, 48:4881–4902]**

- The two-component quark fields are given by

$$\xi(y) = \sum_{\lambda} \chi_{\lambda} \int \frac{[dk]}{\sqrt{2(2\pi)^3}} [b_{\lambda}(k) e^{-ik\cdot y} + d^{\dagger}_{-\lambda}(k) e^{ik\cdot y}],$$

$$\eta(y) = \left(\frac{1}{i\partial^{+}}\right) \left[\sigma^{\perp} \cdot \left(i\partial^{\perp} + gA^{\perp}(y)\right) + im\right] \xi(y).$$

- The dynamical components of the gluon field are given by

$$A^{\perp}(y) = \sum_{\lambda} \int \frac{[dk]}{\sqrt{2(2\pi)^3 k^{+}}} [\epsilon^{\perp}_{\lambda} a_{\lambda}(k) e^{-ik\cdot y} + \epsilon^{\perp *}_{\lambda} a^{\dagger}_{\lambda}(k) e^{ik\cdot y}],$$

where $[dk] = \frac{dk^{+} d^2 \boldsymbol{k}^{\perp}}{\sqrt{2(2\pi)^3 k^{+}}}$, χ_{λ} is the eigenstate of σ^3 and ϵ^{i}_{λ} is the polarization vector of transverse gauge field.

[A. Harindranath, R. Kundu, and W M. Zhang (1999). *Phys. Rev. D*, 59:094013]

- Here, we have suppressed the colour indices.

- The symmetric QCD EMT is defined as

$$
\begin{aligned}
T^{\mu\nu} &= T_q^{\mu\nu} + T_G^{\mu\nu}, \\
T_Q^{\mu\nu} &= \frac{1}{2}\overline{\psi}\, i\left[\gamma^\mu D^\nu + \gamma^\nu D^\mu\right]\psi - \underbrace{g^{\mu\nu}\overline{\psi}\left(i\gamma^\lambda D_\lambda - m\right)\psi}_{\text{(zero by EOM)}}, \\
T_G^{\mu\nu} &= -F_a^{\mu\lambda} F^\nu_{\lambda a} + \frac{1}{4} g^{\mu\nu}\left(F_{\lambda\sigma a}\right)^2 .
\end{aligned}
$$

- The covariant derivative is $iD^\mu = i\overleftrightarrow{\partial}^\mu + gA^\mu$ and $\alpha(i\overleftrightarrow{\partial}^\mu)\beta = \frac{i}{2}\alpha\left(\partial^\mu\beta\right) - \frac{i}{2}\left(\partial^\mu\alpha\right)\beta$.
- The field strength tensor for non-Abelian gauge theory is

$$
F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu ,
$$

where ψ and A^μ are the fermion and boson fields, respectively.
- In the Drell-Yan Frame:

$$
a^\mu = \left(a^+, \boldsymbol{a}^\perp, a^-\right),
$$

$$
p^\mu = \left(p^+, \mathbf{0}^\perp, \frac{m^2}{p^+}\right),\ p'^\mu = \left(p^+, \boldsymbol{\Delta}^\perp, \frac{\boldsymbol{\Delta}^{\perp 2} + m^2}{p^+}\right),\ \Delta^\mu = \left(0, \boldsymbol{\Delta}^\perp, \frac{\boldsymbol{\Delta}^{\perp 2}}{p^+}\right).
$$

Extraction of gravitational form factors

- Define

$$\mathcal{M}^{\mu\nu}_{\lambda\lambda'} = \frac{1}{2}\left[\langle p', \lambda' | T^{\mu\nu}_i(0) | p, \lambda\rangle\right]$$

where the Lorentz indices $(\mu, \nu) \equiv \{+, -, 1, 2\}$, $(\lambda, \lambda') \equiv \{\uparrow, \downarrow\}$ is the helicity of the initial and final state. $\uparrow$ $(\downarrow)$ positive (negative) spin projection along $z-$ axis, $(i = q, G)$.

- The GFFs $A_i(\Delta^2)$ and $B_i(\Delta^2)$ can be extracted from the following equations

$$\mathcal{M}^{++}_{\uparrow\uparrow} + \mathcal{M}^{++}_{\downarrow\downarrow} = 2\,(P^+)^2 A_i(\Delta^2),$$

$$\mathcal{M}^{++}_{\uparrow\downarrow} + \mathcal{M}^{++}_{\downarrow\uparrow} = \frac{i\Delta^{(2)}}{M}\,(P^+)^2 B_i(\Delta^2).$$

- The GFFs $C_q(\Delta^2)$ and $\overline{C}_q(\Delta^2)$ can be extracted from the following equations

$$\mathcal{M}^{11}_{\uparrow\downarrow} + \mathcal{M}^{11}_{\downarrow\uparrow} - \mathcal{M}^{22}_{\uparrow\downarrow} - \mathcal{M}^{22}_{\downarrow\uparrow} = i\left[\frac{B_q(\Delta^2)}{4M} - \frac{C_q(\Delta^2)}{M}\right]\left((\Delta^{(1)})^2\,\Delta^{(2)} - (\Delta^{(2)})^3\right),$$

$$\mathcal{M}^{11}_{\uparrow\downarrow} + \mathcal{M}^{11}_{\downarrow\uparrow} + \mathcal{M}^{22}_{\uparrow\downarrow} + \mathcal{M}^{22}_{\downarrow\uparrow} = i\left[B_q(\Delta^2)\frac{\Delta^2}{4M} - C_q(\Delta^2)\frac{3\Delta^2}{M} + \overline{C}_q(\Delta^2)2M\right]\Delta^{(2)}.$$

- The GFF $C_G(\Delta^2)$, i.e., the D-term, can be extracted from

$$\mathcal{M}^{11}_{\uparrow\downarrow}+\mathcal{M}^{11}_{\downarrow\uparrow}+\mathcal{M}^{22}_{\uparrow\downarrow}+\mathcal{M}^{22}_{\downarrow\uparrow}=i\left[B_G(\Delta^2)\frac{\Delta^2}{4M}-C_G(\Delta^2)\frac{3\Delta^2}{M}+\overline{C}_G(\Delta^2)2M\right]\Delta^{(2)}.$$

- For GFF $\overline{C}_G(\Delta^2)$, we have:

$$\Delta_\mu\mathcal{M}^{\mu(1)}_{\uparrow\downarrow}+\Delta_\mu\mathcal{M}^{\mu(1)}_{\downarrow\uparrow}=-i\Delta^{(1)}\Delta^{(2)}m\overline{C}_G(\Delta^2).$$

- We can write the matrix elements of the EMT in terms of LFWFs:

$$\mathcal{M}^{\mu\nu}_{\lambda,\lambda'}=\sum_{\lambda_1\lambda_2\lambda_1'}\int[dn]\left[\phi^{*\lambda'}_{\lambda_1,\lambda_2}(x,\boldsymbol{\kappa}'^{\perp})\chi^{\dagger}_{\lambda_1'}\mathcal{O}^{\mu\nu}\chi_{\lambda_1'}\phi^{\lambda}_{\lambda_1',\lambda_2}(x,\boldsymbol{\kappa}^{\perp})\right],$$

where $[dn]=\frac{dx\ d^2\boldsymbol{\kappa}^{\perp}}{8\pi^3}$.

- Expressions for quark GFFs are:

$$
\begin{aligned}
A_q(\Delta^2) =& 1 + \frac{g^2\, C_F}{2\pi^2}\left[\frac{11}{10} - \frac{4}{5}\left(1 + \frac{2m^2}{\Delta^2}\right)\frac{f_2}{f_1} - \frac{1}{3}\log\left(\frac{\Lambda^2}{m^2}\right)\right], \\
B_q(\Delta^2) =& \frac{g^2 C_F}{12\pi^2}\,\frac{m^2}{\Delta^2}\,\frac{f_2}{f_1}, \\
D_q(\Delta^2) =& \frac{5g^2 C_F}{6\pi^2}\,\frac{m^2}{\Delta^2}\left(1 - f_1 f_2\right) = 4\, C_q(\Delta^2), \\
\overline{C}_q(\Delta^2) =& \frac{g^2 C_F}{72\pi^2}\left(29 - 30\, f_1\, f_2 + 3\log\left(\frac{\Lambda^2}{m^2}\right)\right),
\end{aligned}
$$

where

$$
f_1 = \frac{1}{2}\sqrt{1 + \frac{4m^2}{\Delta^2}}, \quad f_2 = \log\left(1 + \frac{\Delta^2\,(1 + 2f_1)}{2m^2}\right),
$$

C_F is the colour factor and Λ is the ultra-violet cut-off.

[J. More, A. Mukherjee, S. Nair, and S. Saha (2022). *Phys. Rev. D*, 105(5):056017]

- Expressions for gluon GFFs are:

$$A_G(\Delta^2) = \frac{g^2 C_F}{8\pi^2}\left[\frac{29}{9} + \frac{4}{3}\ln\left(\frac{\Lambda^2}{m^2}\right) - \int dx \left(\left(1+(1-x)^2\right) + \frac{4m^2x^2}{\Delta^2(1-x)}\right)\frac{\tilde{f}_2}{\tilde{f}_1}\right],$$

$$B_G(\Delta^2) = -\frac{g^2 C_F}{2\pi^2}\int dx\, \frac{m^2x^2}{\Delta^2}\frac{\tilde{f}_2}{\tilde{f}_1},$$

$$D_G(\Delta^2) = \frac{g^2 C_F}{6\pi^2}\left[\frac{2m^2}{3\Delta^2} + \int dx\, \frac{m^2}{\Delta^4}\left(x\left((2-x)\Delta^2 - 4m^2x\right)\right)\right]\frac{\tilde{f}_2}{\tilde{f}_1},$$

$$\overline{C}_G(\Delta^2) = \frac{g^2 C_F}{72\pi^2}\left[10 + 9\int dx\, \left(x - \frac{4m^2x^2}{\Delta^2(1-x)}\right)\frac{\tilde{f}_2}{\tilde{f}_1} - 3\ln\left(\frac{\Lambda^2}{m^2}\right)\right],$$

where

$$\tilde{f}_1 = \sqrt{1 + \frac{4m^2x^2}{\Delta^2(1-x)^2}},$$

$$\tilde{f}_2 = ln\left(\frac{1+\tilde{f}_1}{-1+\tilde{f}_1}\right),$$

C_F is the colour factor and Λ is the ultra-violet cut-off.

[J. More, A. Mukherjee, S. Nair, and S. Saha (2023). *Phys. Rev. D*, 107(11):116005]

GFFs: $A(\Delta^2)$ and $B(\Delta^2)$

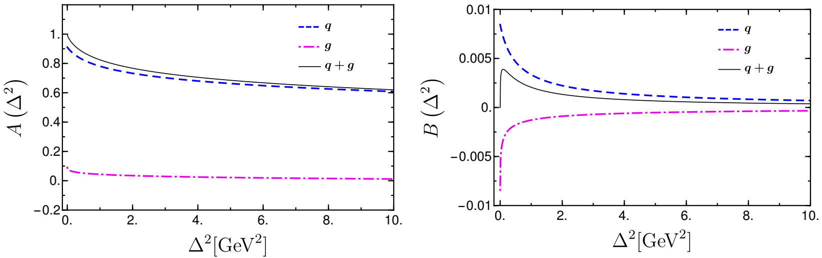


Figure: 1. Plots of the GFF $A(\Delta^2)$ and $B(\Delta^2)$ as a function of Δ^2. The dashed blue curve and the dot-dashed magenta curve are for the quark (q) and gluon (g) form factors respectively. The solid black curve is for the sum of quark and gluon ($q+g$) contribution. Here $m = 0.3$ GeV and $\Lambda = 2$ GeV.

$$\sum_{(i=q,G)} A_i(0) = 1, \qquad \sum_{(i=q,G)} B_i(0) = 0.$$

- Dressed electron in QED: **[S J. Brodsky, D S. Hwang, B Q. Ma, and I. Schmidt (2001). *Nucl. Phys. B*, 593:311–335]**

GFF: $D(\Delta^2)$

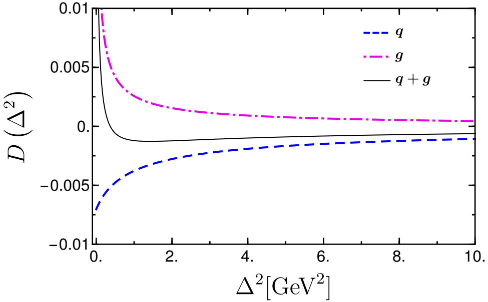


Figure: 2. Plots of the GFF $D(\Delta^2)$ as a function of Δ^2. The dashed blue curve and the dot-dashed magenta curve are for the quark (q) and gluon (g) form factors respectively. The solid black curve is for the sum of quark and gluon ($q+g$) contribution. Here $m = 0.3$ GeV and $\Lambda = 2$ GeV.

Comparison of electron D-term with Metz *et al.*

MPR: [A. Metz, B. Pasquini, and S. Rodini (2021). *Phys. Lett. B*, 820:136501]

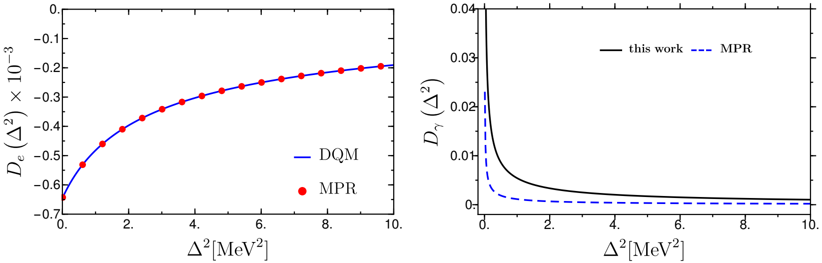


Figure: 3. Plot of photon GFF $D_\gamma(\Delta^2)$ and $D_e(\Delta^2)$ as function of Δ^2. Here, we set $m = 0.511$ MeV, $\alpha = \frac{1}{137}$.

- The D-term for the electron dressed with a photon has been calculated by Metz *et al.* using the Feynman diagram method. We have calculated using the Light-Front Wave Function (LFWF) in the QED limit.

GFF: $\overline{C}(\Delta^2)$

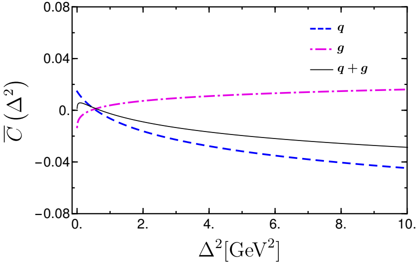


Figure: 2. Plots of the GFF $\overline{C}(\Delta^2)$ as a function of Δ^2. The dashed blue curve and the dot-dashed magenta curve are for the quark (q) and gluon (g) form factors respectively. The solid black curve is for the sum of quark and gluon ($q + g$) contribution. Here $m = 0.3$ GeV and $\Lambda = 2$ GeV.

- $$\sum_{(i=q,G)} \overline{C}_i(\Delta^2) = 0 \quad \text{(Expected!)}$$
  $$\sum_{(i=q,G)} \overline{C}_i(0) = 0 \quad \text{(Our case)}$$

- Non-diagonal matrix element:

$$2g\left(\partial^i A_a^j\right)\frac{1}{\partial^+}\left(\xi^\dagger T^a \xi\right)$$

(Zero modes: $k^+ = 0$)

Pressure and Shear Force Distributions

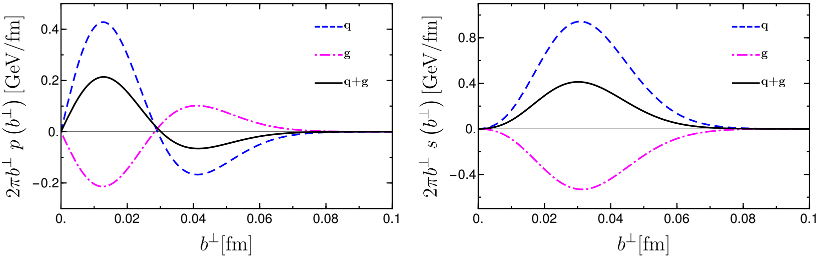


Figure: 4. Plots of the pressure distribution $2\pi b^{\perp}\ p(b^{\perp})$ and the shear force distribution $2\pi b^{\perp}\ s(b^{\perp})$ as a function of $b^{\perp}$. The dashed blue curve and the dot-dashed magenta curve are for the quark (q) and gluon (g) contributions respectively. The solid black curve is for the sum of quark and gluon ($q+g$) contribution. Here $\sigma = 0.2$.

- Von Laue stability condition:

$$\int d^2\boldsymbol{b}^{\perp}\ p(\boldsymbol{b}^{\perp}) = 0.$$

[A. Freese and G A. Miller (2021). *Phys. Rev. D*, 103:094023]

Normal and Tangential Force Distributions

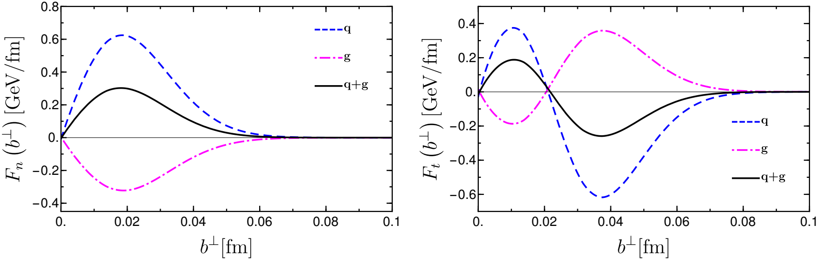


Figure: 5. Plots of the normal force $F_n(b^\perp)$, and the tangential force $F_t(b^\perp)$ as a function of $b^\perp$. The dashed blue curve and the dot-dashed magenta curve are for the quark (q) and gluon (g) contributions respectively. The solid black curve is for the sum of quark and gluon ($q+g$) contributions. Here $\sigma = 0.2$.

Two-Dimensional Energy Density

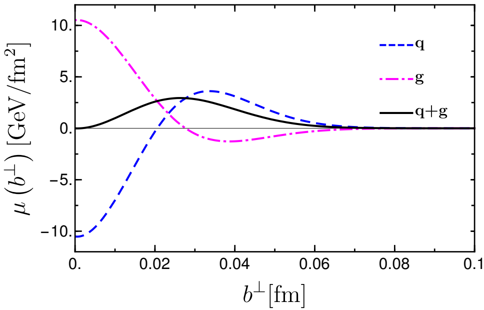


Figure: 6. Plot of the energy density $\mu(b^{\perp})$ as a function of $b^{\perp}$. The dashed blue curve and the dot-dashed magenta curve are for the quark (q) and gluon (g) contributions respectively. The solid black curve is for the sum of quark and gluon ($q+g$) contribution. Here $\sigma = 0.2$.

$$\mu_i(\boldsymbol{b}^{\perp}) = m\left[\frac{1}{2}A_i(\boldsymbol{b}^{\perp}) + \overline{C}_i(\boldsymbol{b}^{\perp}) + \frac{1}{4m^2}\frac{1}{\boldsymbol{b}^{\perp}}\frac{d}{d\boldsymbol{b}^{\perp}}\left(\boldsymbol{b}^{\perp}\frac{d}{d\boldsymbol{b}^{\perp}}\left[\frac{1}{2}B_i(\boldsymbol{b}^{\perp}) - 4C_i(\boldsymbol{b}^{\perp})\right]\right)\right]$$

[C. Lorcé, H. Moutarde, and A P. Trawiński (2019). *Eur. Phys. J. C*, 79(1):89]

Conclusions

- We discussed the mechanical properties of nucleons through the matrix elements of the energy-momentum tensor.
- We use the light-front coordinates and the two-component light front QCD with light front gauge $A^+ = 0$.
- We have studied the Gravitational Form Factors (GFFs), which originate from a symmetric Energy-Momentum Tensor (EMT), for a composite spin-$\frac{1}{2}$ state i.e., the dressed quark state.
- This state includes the quark-gluon interaction at one loop in QCD which enables us to study the GFFs in the presence of interaction.
- We have also analysed the mechanical properties like pressure and shear stress distributions in this state which gave an intuitive picture of the spatial distributions in a two-particle relativistic composite state.

Thank you!

Back ups

- The differential cross-section

$$d\sigma \propto \left| \langle k_f, s_f | j^\mu(0) | k_i, s_i \rangle \frac{1}{Q^2} \langle X | J_\mu(0) | P, S \rangle \right|^2 \approx L^e_{\mu\nu} W^{\mu\nu},$$

where $j^\mu = e\overline{\psi}\gamma^\mu\psi$, with ψ as the lepton spinor and e is the electric charge, J^μ is the hadron current and $|X\rangle$ is the final state with momentum P_X.

- The lepton and the hadron tensor can be written as

$$L^e_{\mu\nu} = 2\left(k_f k_i^\nu + k_i^\mu k_f^\nu - \left(k_f \cdot k_i - m^2\right) g^{\mu\nu}\right)$$

$$W^{\mu\nu} = W_1(\nu, Q^2)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{W_2(\nu, Q^2)}{M_p^2}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right),$$

- In the laboratory frame, neglecting the proton mass the cross-section can be written as

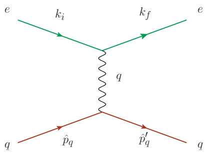
$$\frac{d\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 + (1-y)^2\right) F_1(x, Q^2) + \frac{(1-y)}{x}\left(F_2(x, Q^2) - 2xF_1(x, Q^2)\right)\right],$$

where, $x = \frac{Q^2}{2M_p\nu}$, $y = \left.\frac{p_i\cdot q}{p_i\cdot k}\right|_{\text{lab}} = \frac{\nu}{E}$, $\nu = E - E'$, $F_1 = M_p W_1$, $F_2 = \nu W_2 = EyW_2$. The kinematic region of $ep \to eX$ is $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

- The functions $F_i(x, Q^2)'s$ are called structure functions which parametrize the structure of the target as seen by the virtual photon are functions of x and Q^2 .

- The point-like constituents also move parallel with the proton and carry a fraction ξ of its momentum, i.e. $\hat{p}_q^\mu = \xi p_i^\mu$.
  - Applying the mass-shell condition for the outgoing quark we have

$$\hat{p}_q'^2 = (p_q + q)^2 \implies 2\xi p_i \cdot q - Q^2 = 0 \implies x = \xi.$$



- The cross-section for this electron-quark scattering is

$$\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[1 + (1-y)^2\right]\frac{1}{2}e_q^2\delta(x-\xi).$$

Compare with

$$\frac{d\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1+(1-y)^2\right)F_1(x,Q^2) + \frac{(1-y)}{x}\left(F_2(x,Q^2) - 2xF_1(x,Q^2)\right)\right],$$

- The structure functions in this simple model are

$$\hat{F}_2 = 2x\hat{F}_1 = xe_q^2\delta(x-\xi).$$

- This result suggests that the virtual photon probes a constituent quark with momentum fraction $\xi = x$.

- The quarks carry a range of momentum fractions. We can assign a PDF $f^q(\xi)d\xi$ to find a quark q with momentum fraction between ξ and $\xi + d\xi$, where $0 \leq \xi \leq 1$.

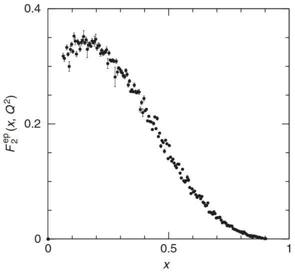


Figure: Measurement of $F_2^{ep}(x, Q^2)$ as a function of x for deep inelastic scattering events with $2 < Q^2/\mathrm{GeV}^2 < 30$ at SLAC, [L. W. Whitlow et. al. (1979). *Phys. Rev. D*, 20:1471-1552], [A. Bodek et al. (1979). *Phys. Rev. D*, 20:1471–1552]. Image source: [M. Thomson (2013). Cambridge University Press, New York].

- An important assumption of electron-proton DIS is that the virtual photon scatters incoherently off the individual quarks.

$$\text{Callan-Gross relation: } F_2(x) = 2xF_1(x) = \sum_q \int_0^1 d\xi f^q(\xi) x e_q^2 \delta(x - \xi) = \sum_q e_q^2\, x f^q(x).$$

- This is known as the 'naive parton model' [R. P. Feynman (1973), Photon-hadron interaction].

- The $RR \to RR$ and $LL \to LL$ configurations occur where the total angular momentum in the $z-$direction is zero, where z is the direction along the incoming electron.

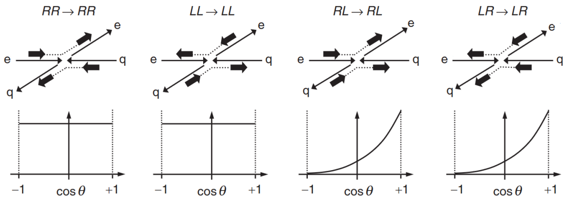


Figure: Helicity combinations contributing to the $e\,q \to e\,q$ process in ultra-relativistic limit. Image source: [M. Thomson (2013). Cambridge University Press, New York].

- The $RL \to RL$ and $LR \to LR$ configurations transition from $J_z = \pm 1$ to $J_{z'} = \pm 1$, where z' is along the outgoing electron. This depends on $d^j_{\lambda,\lambda'}(\theta) = \langle j, \lambda' | e^{-i\theta J_y} | j, \lambda \rangle$, with y perpendicular to the interaction plane and θ the center-of-mass scattering angle. In COM frame

$$d^1_{1,1}(\theta) = d^1_{-1,-1}(\theta) = \frac{1}{2}\,(1 + \cos\theta) = (1 - y)\,,$$

$$\text{Electron-quark cross-section: } \frac{d^2\hat{\sigma}}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\underbrace{1}_{\text{(same helicity)}} + \underbrace{(1-y)^2}_{\text{(opp. helicity)}}\right]\frac{1}{2}e_q^2\,\delta(x - \xi).$$

- Experimental measurement of this relation:

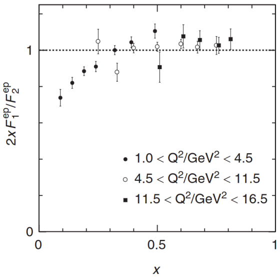
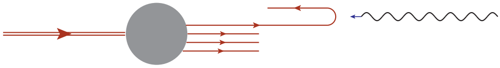


Figure: Experimental measurement of Callan-Gross relation at SLAC, [L. W. Whitlow et. al. (1979). *Phys. Rev. D*, 20:1471-1552], [A. Bodek et al. (1979). *Phys. Rev. D*, 20:1471–1552]. Image source: [M. Thomson (2013). Cambridge University Press, New York].

- The Callan-Gross relation follows from the fact that spin$-\frac{1}{2}$ quarks can only absorb a transversely polarized virtual photon.
- This head-on collision of electron and quark is mediated by a photon of spin-1. This can be explained by helicity conservation only if the quark has spin $\frac{1}{2}$.



- A proton target with its charge quark flavours, we can write

$$F_2(x) = x\left[\frac{4}{9}\left(u(x)+\overline{u}(x)\right)+\frac{1}{9}\left(d(x)+\overline{d}(x)+s(x)+\overline{s}(x)\right)\right],$$

  where $q(x)$, $\overline{q}(x)$, $\{q(x) = u(x),\ d(x),\ s(x)\}$ are the PDFs of different flavours of quarks and anti-quarks within the proton.

- PDFs cannot be derived from first principles because the QCD coupling is large, $\alpha_S \sim \mathcal{O}(1)$, making perturbation theory inapplicable. PDFs can only be extracted from experiments.

  [A D. Martin, W J. Stirling, and R G. Roberts (1994). *Phys. Rev. D*, 50:6734–6752]

- The proton consists of three valence quarks (uud) with electric charge and baryon quantum number, plus an infinite sea of light $q\overline{q}$ pairs. Assuming the sea is symmetric in all quark flavours at a scale of $\mathcal{O}(1\text{ GeV})$, we would have

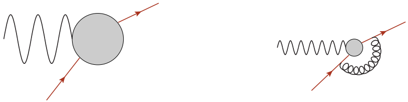
$$u(x) = u_v(x) + S(x), \quad d(x) = d_v(x) + S(x),$$
$$S(x) = u_s(x) = \overline{u}_s(x) = d_s(x) = \overline{d}(x) = s_s(x) = \overline{s}_s(x),$$

  with the sum rules

$$\int_0^1 dx\, u_V(x) = 2, \quad \int_0^1 dx\, d_V(x) = 1, \quad \int_0^1 dx\, x\left[u_V(x) + d_V(x) + 6S(x)\right] \approx 0.5.$$

- Quarks carry 50% of a proton's momentum, with the rest due to gluons, which are seen in high-energy processes like large transverse momentum jets and heavy quark production.

- At low Q^2 the virtual photon has a long wavelength, so resolving any sub-structure below a certain length scale is impossible.



- Whereas, at high Q^2, due to the shorter wavelength of the virtual photon it can resolve finer details.
- Gluon radiation $(\gamma^* q \to qg)$ cross-section to the parton model cross-section

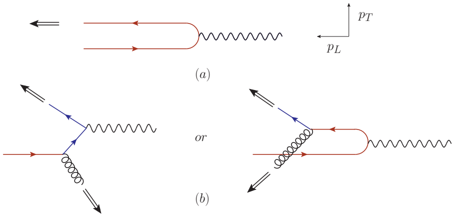


Figure: (a) Parton model diagram without transverse momentum. (b) Gluon radiation diagrams with non-zero transverse momentum jets.

- The QCD modified structure function is given by

$$\left.\frac{F_2(x,Q^2)}{x}\right|_{(\gamma^* q\to qg)} = \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi} q_0(\xi)\left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P_{qq}\left(\frac{x}{\xi}\right) \underbrace{\log\frac{Q^2}{\kappa^2}}_{\text{(gluon emission)}}\right],$$

  where $q_0(\xi)$ is the bare quark PDF and κ is cut-off on the transverse momentum to regularize the divergence at $p_T^2 \to 0$, which is called *collinear divergence*.
- The splitting function $P_{qq}(z)$ has the following form

$$P_{qq}(z) = \frac{4}{3}\left(\frac{1+z^2}{1-z}\right),$$

  and represents the probability of a quark emitting a gluon and becoming a quark with momentum reduced by a fraction of z.
- The limit $p_T^2 \to 0$, when the gluon is emitted parallel to the quark, corresponds to a long-range (soft) part of the strong interaction which is not calculable from perturbation theory.

- We absorb the collinear singularities into the bare distribution at a *factorization scale* μ and define a modified quark PDF

$$q\left(x,\mu^2\right) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_0(\xi) P_{qq}\left(\frac{x}{\xi}\right) \log\frac{\mu^2}{\kappa^2}.$$

- Define the point-like form factor as

$$\frac{F_2(x,Q^2)}{x} = \sum_q e_q^2 \left(q(x,\mu^2) + \Delta q(x,Q^2,\mu^2)\right),$$

where

$$\Delta q(x,Q^2,\mu^2) = \frac{\alpha_s}{2\pi}\log\left(\frac{Q^2}{\mu^2}\right)\int_x^1 \frac{d\xi}{\xi}\, q(\xi,\mu^2) P_{qq}\left(\frac{x}{\xi}\right),$$

this indicated the quark PDFs are now dependent on Q^2 and also the factorization scale μ.

- The ability to separate short- and long-distance (non-perturbative) contributions to structure functions is a fundamental property called *Collinear factorization*.
- Altarelli-Parisi evolution equation

$$\frac{d}{d\log Q^2} q(x,Q^2,\mu^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\, q(\xi,\mu^2) P_{qq}\left(\frac{x}{\xi}\right).$$

A quark with momentum fraction x may come from a quark with fraction ξ that radiated a gluon, with probability proportional to $\alpha_s P_{qq}(x/\xi)$.

- A gluon-initiated process where a gluon produces a quark-antiquark pair to which the virtual photon couples

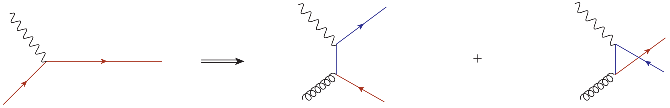


Figure: $\mathcal{O}(\alpha\alpha_s)$ contributions ($\gamma^* g \to q\bar{q}$) to DIS.

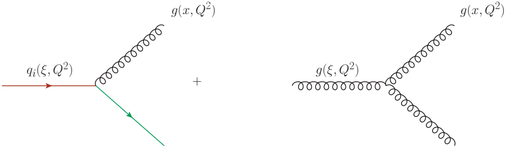
$$\left.\frac{F_2(x,Q^2)}{x}\right|_{(\gamma^* g \to q\bar{q})} = \sum_q e_q^2 \int_x^1 \frac{d\xi}{\xi}\, g(\xi,\mu^2)\, \frac{\alpha_s}{2\pi} P_{qg}\left(\frac{x}{\xi}\right) \log\left(\frac{Q^2}{\mu^2}\right),$$

where $g(\xi,\mu^2)$ is gluon PDF and the splitting function is given by

$$P_{qg}(z) = \frac{1}{2}\left[z^2 + (1-z)^2\right],$$

represents the probability of finding a $q\bar{q}$ pair with momentum fraction z of the momentum of the gluon from which the pair has created.

- Evolution of gluon PDF



$$\frac{d}{d\log Q^2} g(x, Q^2, \mu^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \left[\sum_i q_i(\xi, Q^2, \mu^2) P_{gq}\left(\frac{x}{\xi}\right) + g(\xi, Q^2, \mu^2) P_{gg}\left(\frac{x}{\xi}\right) \right]$$

where the sum $i = 1 \ldots 2n_f$ runs overall flavours of quark and anti-quark flavours and the splitting functions are given by

$$P_{gq}(x) = \frac{4}{3} \frac{1 + (1-z)^2}{z},$$
$$P_{gg}(x) = 6\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right].$$

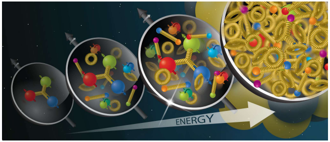


Image source: BNL photo albums

- The LFQCD Hamiltonian can be written in terms of these two-component fields in the following way

$$H = \int dx^- d^2\boldsymbol{x}^\perp \; (\mathcal{H}_0 + \mathcal{H}_{\rm int}) \, ,$$

where

$$\begin{aligned}\mathcal{H}_0 =& \frac{1}{2}\left(\partial^i A_a^j\right)\left(\partial^i A_a^j\right) + \xi^\dagger \left(\frac{-\partial^{\perp 2} + m^2}{i\partial^+}\right)\xi, \\ \mathcal{H}_{\rm int} =& \mathcal{H}_{qqg} + \mathcal{H}_{ggg} + \mathcal{H}_{qqgg} + \mathcal{H}_{qqqq} + \mathcal{H}_{gggg},\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_{qqg} =& g\xi^\dagger \Big[-2\left(\frac{1}{\partial^+}\right)\left(\partial^\perp \cdot \boldsymbol{A}^\perp\right) + \boldsymbol{\sigma}^\perp \cdot \boldsymbol{A}^\perp \left(\frac{1}{\partial^+}\right)\left(\boldsymbol{\sigma}^\perp \cdot \partial^\perp + m\right) \\ & + \left(\frac{1}{\partial^+}\right)\left(\boldsymbol{\sigma}^\perp \cdot \partial^\perp - m\right)\boldsymbol{\sigma}^\perp \cdot \boldsymbol{A}^\perp \Big]\xi, \\ \mathcal{H}_{ggg} =& g f^{abc}\left[\partial^i A_a^j A_b^i A_c^j + \left(\partial^\perp \cdot \boldsymbol{A}^\perp\right)\left(\frac{1}{\partial^+}\right)\left(A_b^j \partial^+ A_c^j\right)\right], \\ \mathcal{H}_{qqgg} =& g^2 \Big[\xi^\dagger \left(\boldsymbol{\sigma}^\perp \cdot \boldsymbol{A}^\perp\right)\left(\frac{1}{i\partial^+}\right)\left(\boldsymbol{\sigma}^\perp \cdot \boldsymbol{A}^\perp\right)\xi \\ & + 2\left(\frac{1}{\partial^+}\right)\left(f^{abc} A_b^i \partial^+ A_c^i\right)\left(\frac{1}{\partial^+}\right)\left(\xi^\dagger t^a \xi\right)\Big] \\ =& \mathcal{H}_{qqgg1} + \mathcal{H}_{qqgg2},\end{aligned}$$

[W M. Zhang and A. Harindranath (1993). *Phys. Rev. D*, 48:4881–4902]

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\mathcal{H}_{\text{int}} =& \mathcal{H}_{qqg} + \mathcal{H}_{ggg} + \mathcal{H}_{qqgg} + \mathcal{H}_{qqqq} + \mathcal{H}_{gggg},
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{qqqq} &= 2g^2 \left[\left(\frac{1}{\partial^+}\right)\left(\xi^\dagger t^a \xi\right)\left(\frac{1}{\partial^+}\right)\left(\xi^\dagger t^a \xi\right)\right], \\
\mathcal{H}_{gggg} &= \frac{g^2}{4} f^{abc} f^{ade} \Big[A_b^i A_c^j A_d^i A_e^j \\
&\quad + 2\left(\frac{1}{\partial^+}\right)\left(A_b^i \partial^+ A_c^i\right)\left(\frac{1}{\partial^+}\right)\left(A_d^j \partial^+ A_e^j\right)\Big], \\
&= \mathcal{H}_{gggg1} + \mathcal{H}_{gggg2}.
\end{aligned}$$

[W M. Zhang and A. Harindranath (1993). *Phys. Rev. D*, 48:4881–4902]

- The matrix elements of the energy-momentum tensor (EMT) of QCD are essential for addressing these questions.

$$T_q^{\mu\nu} = \frac{1}{2}\overline{\psi}_q \left[\gamma^\mu iD^\nu + \gamma^\nu iD^\mu\right] \psi_q,$$
$$T_G^{\mu\nu} = -F^{a\mu\lambda}F^{a\nu}_\lambda + \frac{1}{4}g^{\mu\nu}\left(F^{a\lambda\sigma}\right)^2 - g^{\mu\nu}\overline{\psi}_q\left(i\gamma^\lambda D_\lambda - m\right)\psi_q,$$

Covariant derivative: $iD^\mu = i\overleftrightarrow{\partial}^\mu + gA^\mu, \alpha\left(i\overleftrightarrow{\partial}^\mu\right)\beta = \frac{i}{2}\alpha\left(\partial^\mu\beta\right) - \frac{i}{2}\left(\partial^\mu\alpha\right)\beta,$
$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^{nu}A_a^\mu + gf_{abc}A_b^\mu A_c^\nu.$

- Parameterizing the EMT for a spin-$\frac{1}{2}$ system can be achieved in terms of four gravitational form factors (GFFs) **[M V. Polyakov and P. Schweitzer (2018).** ***Int. J. Mod. Phys. A*****, 33(26):1830025]**

$$\begin{aligned}&\langle p',\lambda'|T_i^{\mu\nu}(0)|p,\lambda\rangle\\ =&\overline{u}(p',\lambda')\Bigg[-B_i(\Delta^2)\frac{P^\mu P^\nu}{M} + \left(A_i(\Delta^2)+B_i(\Delta^2)\right)\frac{1}{2}\left(\gamma^\mu P^\nu + \gamma^\nu P^\mu\right)\\ &+C_i(\Delta^2)\frac{\Delta^\mu\Delta^\nu - \Delta^2 g^{\mu\nu}}{M} + \overline{C}_i(\Delta^2)Mg^{\mu\nu}\Bigg]u(p,\lambda),\end{aligned}$$

- Gordon decomposition:

$$\overline{u}(p',\lambda)\left[\overline{P}^\mu i\sigma^{\nu\lambda}\Delta_\lambda + \overline{P}^\nu i\sigma^{\mu\lambda}\Delta_\lambda\right]u(p,\lambda) = \overline{u}(p',\lambda)\left[2M\left(\gamma^\mu\overline{P}^\nu + \gamma^\nu\overline{P}^\mu\right) - 4\overline{P}^\mu\overline{P}^\nu\right]u(p,\lambda).$$

- Conservation of total energy-momentum tensor (EMT) rules out terms like $(\overline{P}^\mu\Delta^\nu + \Delta^\mu\overline{P}^\nu)$ and $(\gamma^\mu\Delta^\nu + \gamma^\nu\Delta^\mu)$ also puts a constraint $\sum_i \overline{C}_i(\Delta^2) = 0$.

- In the usual Hamiltonian dynamics, one works with a set of generalized coordinates with initial conditions defined at some instant of time, preferably at $x^0 = 0$.

$$
\begin{aligned}
g^{00} &= 1,\ g^{ii} = -1,\ i = (1, 2, 3) \\
g^{\mu\nu} &= 0,\ (\mu \neq \nu),
\end{aligned}
$$

- one can parameterize the space-time by a functional relation

$$\tilde{x}^{\nu} = \tilde{x}^{\nu}(x^{\mu}),$$

provided the inverse $x^{\mu}(\tilde{x}^{\nu})$ also exists as well.

- The transformation of coordinates conserves the arc length, i.e. $(ds)^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = \tilde{g}_{\mu\nu} d\tilde{x}^{\mu} d\tilde{x}^{\nu}$.

$$\tilde{g}_{\eta\lambda} = \left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\eta}} \right) g_{\mu\nu} \left(\frac{\partial x^{\mu}}{\partial \tilde{x}^{\lambda}} \right)$$

- The four-volume elements of the two parametrizations are related by the Jacobian $\mathcal{J}(\tilde{x})$ with $d^4 x = \mathcal{J}(\tilde{x}) d^4 \tilde{x}$.

- The Poincaré algebra can be written in terms of four vector P^μ and generalized angular momentum $M^{\mu\nu}$:

$$\begin{aligned}\left[P^\mu, P^\nu\right] =& 0, \\ \left[M^{\mu\nu}, P^\rho\right] =& i\left(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu\right), \\ \left[M^{\mu\nu}, M^{\rho\sigma}\right] =& i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right),\end{aligned}$$

  where P^μ are the generators of translation and $M^{\mu\nu}$ are the generators of the Lorentz group.

- The rotation and the boost operator are defined as $M_{ij} = \epsilon_{ijk}J^k$ and $M^{0i} = K^i$ respectively which obey the commutation relations given by

$$\left[J^i, J^j\right] = i\epsilon_{ijk}J^k, \quad \left[J^i, K^j\right] = i\epsilon_{ijk}K^k, \quad \left[K^i, K^j\right] = -i\epsilon_{ijk}J^k$$

- The momentum and the generalized angular momentum tensor can be defined in terms of the energy-momentum tensor of a theory as follows

$$\begin{aligned}P^\mu &= \int d^3x\, T^{0\mu} \\ M^{\mu\nu} &= \int d^3x\, \left[x^\mu T^{0\nu} - x^\nu T^{0\mu}\right].\end{aligned}$$

- The differential cross-section for DIS

$$d\sigma \propto \left| \langle k_f, s_f | j^\mu(0) | k_i, s_i \rangle \frac{1}{Q^2} \langle X | J_\mu(0) | P, S \rangle \right|^2 \approx L^e_{\mu\nu} W^{\mu\nu}.$$

- The hadron tensor can be written as

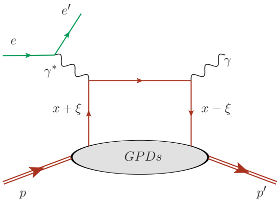
$$\begin{aligned} W^{\mu\nu}(P, S, P_X) =& \frac{1}{4\pi M_p} \sum_X \int \frac{d^3 \boldsymbol{P}_x}{(2\pi)^3 3 P_X^0} \langle P, S | J^\mu(0) | X \rangle \langle X | J^\nu(0) | P, S \rangle (2\pi)^4 \\ & \delta^{(4)}(P + q - P_X), \\ =& \frac{1}{2\pi} \int d^4\xi \, e^{iq\cdot\xi} \langle P, S | \left[J^\mu(\xi), J^\nu(0) \right] | P, S \rangle. \end{aligned}$$

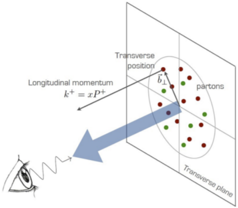
- In the Bjorken limit we have $q^- \to \infty$ and the large oscillations in the exponential $e^{iq\cdot\xi} = e^{i\left(q^+\xi^- + q^-\xi^+\right)}$ make the integral vanish unless $\xi^+ \to 0$.
- Now $\xi^2 \geq 0$, since no space-like separation between the two currents.
- But $\xi^+ \to 0$ imples $\xi^2 = 0$, it is a light-cone separation.

- The generalized parton distribution functions

$$\Phi^{[\Gamma]}(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p'|\overline{\psi}\left(-\frac{1}{2}z\right)\Gamma \mathcal{U}_{[-\frac{z}{2},\frac{z}{2}]}\psi\left(\frac{1}{2}z\right)|p\rangle\Big|_{z^+=0,\boldsymbol{z}^\perp=0},$$

Γ : Dirac matrix and $\mathcal{U}_{[-\frac{z}{2},\frac{z}{2}]}$: Gauge link.





- The gauge link

$$\mathcal{U}_{[-\frac{z}{2},\frac{z}{2}]} = \mathcal{P}\exp\left[-ig\int_{-\frac{z}{2}}^{\frac{z}{2}} d\eta^\mu A_\mu(\eta)\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(-ig)^n \mathcal{P}\int_{-\frac{z}{2}}^{\frac{z}{2}} d\eta_n^{\mu_n}\dots d\eta_1^{\mu_1} A_{\mu_n}(\eta_n)\dots A_{\mu_1}(z_1),$$

$\mathcal{P}$ denotes path ordering from $-\frac{z}{2}$ to $\frac{z}{2}$.

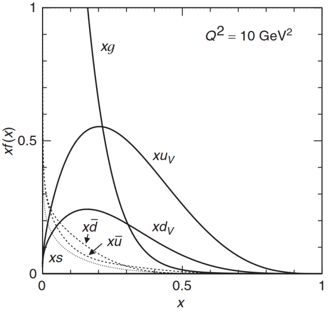
- Exchange of more than two gluons is suppressed except for gluons with longitudinal polarization.
- In light cone gauge $A^+ = 0$, the gauge link becomes unity.

Now suppose the electron is interacting with an external electromagnetic field A^μ, then the term corresponds to the second term will have an amplitude

$$T^{(2)}_{fi} = ie \int d^4x \, \frac{1}{2m} \left(\overline{u}_f i\sigma^{\mu\nu} q_\nu u_i\right) A_\mu \, e^{iq\cdot x} = \frac{ie}{2m} \int d^4x \, \left(\overline{u}_f \sigma_{\mu\nu} u_i\right) \, \frac{1}{2} F^{\mu\nu} e^{iq\cdot x}.$$

Now if we note that $F^{23} = \boldsymbol{B}^{(1)}, \; F^{31} = \boldsymbol{B}^{(2)}, \; F^{12} = \boldsymbol{B}^{(3)}$ and $\sigma^{23} = \sigma^{(1)}, \; \sigma^{31} = \sigma^{(2)}, \; \sigma^{12} = \sigma^{(3)}$, where $\boldsymbol{B}$ is the magnetic field and $\sigma^{(1)}, \; \sigma^{(2)}, \; \sigma^{(3)}$ are Pauli sigma matrices. Then the above equation becomes

$$T^{(2)}_{fi} = \frac{i}{2} \int d^4x \, \overline{u}_f \left(\frac{e}{2m} \boldsymbol{\sigma} \cdot \boldsymbol{B}\right) u_i \, e^{iq\cdot x}. \tag{1}$$



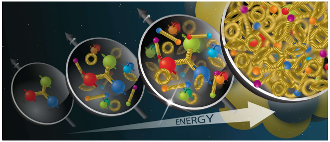
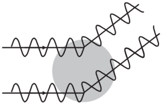


Figure: Left: Different proton PDFs at $Q^2 = 10$ GeV^2. Right: Evolving picture of proton.

Image source: BNL photo albums

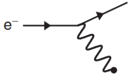
- A form factor accounts for the phase differences in scattered waves from various points on the target.



- At low electron energies, the wavelength of the virtual photon $\lambda >> r_p$. The *ep* scattering can be described as the elastic scattering of the electron from a *static potential* of a proton.

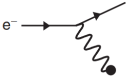
$$J_p^\mu = (Ze\,\rho(\boldsymbol{x}), \mathbf{0})\,,$$

$$F(\boldsymbol{q}) = \int d^3\boldsymbol{x}\;\rho(\boldsymbol{x})e^{i\boldsymbol{q}\cdot\boldsymbol{x}}.$$



$\lambda \gg r_p$

- At higher electron energies, the wavelength of the virtual photon $\lambda \sim r_p$. The *ep* scattering needs to include effects from extended charge as well as magnetic distributions of the proton.



$\lambda \sim r_p$

Image source: [M. Thomson (2013). Cambridge University Press, New York]

Dispersion relation

- Simple dispersion relation: $p^- = \frac{p^{\perp 2}+m^2}{p^+}$.
  - The dependence of light-front energy p^- to the transverse momentum $\boldsymbol{p}^\perp$ is similar to the non-relativistic dispersion relation.
  - With positive energy p^- we will always have positive p^+. This fact has implications for simplifying the vacuum of the theory in light-front.
  - For eg. in instant form

$$\left[a(-\boldsymbol{p}_2)a^\dagger(\boldsymbol{p}_2)\right]\left[a(-\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_1)\right]|0\rangle,$$

    there is a finite overlap between the ground state of the free theory and the interaction theory

$$e^{-iHt}|0\rangle = e^{-iE_0 t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n t}|n\rangle\langle n|0\rangle,$$

    where $|\Omega\rangle$ is the ground state of the Hamiltonian with interaction (H), E_n are the eigenvalues of the H and $E_0 = \langle\Omega|H|\Omega\rangle$.
  - The light-front vacuum is less complicated than the instant form.

Observables for gravitational form factors

- Graviton-proton scattering, as the EMT couples to the graviton.
  [H. Pagels (1966). *Phys. Rev.*, 144:1250–1260]

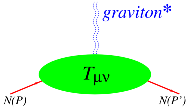
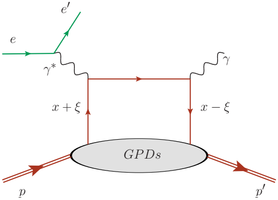


Image source: [M V. Polyakov and P. Schweitzer (2018). *Int. J. Mod. Phys. A*, 33(26):1830025]

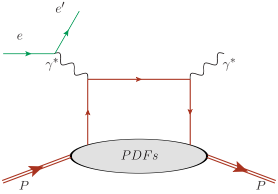
- Indirect methods have been developed to gather information about the matrix elements of the EMT through studies of hard exclusive scatterings e.g. deeply virtual Compton scattering (DVCS).

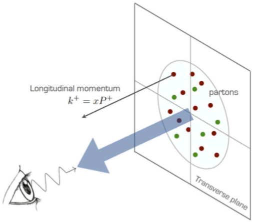


Generalized Parton Distributions

- The parton distribution functions in DIS

$$f_1(x) = \int \frac{d\xi^-}{8\pi} e^{-\frac{i}{2}xP^+\xi^-} \langle P.S|\overline{\psi}^q\left(0, \xi^-, \mathbf{0}^\perp\right)\gamma^+\psi^q(0)|P,S\rangle.$$

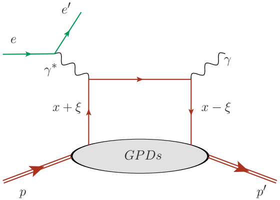


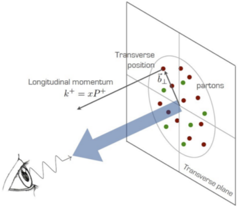


- The generalized parton distribution functions

$$\Phi^{[\Gamma]}(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p'|\overline{\psi}\left(-\frac{1}{2}z\right)\Gamma\psi\left(\frac{1}{2}z\right)|p\rangle\Big|_{z^+=0.\boldsymbol{z}^\perp=0},$$

Γ : Dirac matrix. Finite momentum tansfer in transverse direction $\boldsymbol{\Delta}^\perp \overset{\text{FT}}{\longleftrightarrow} \boldsymbol{b}^\perp$.





- GPDs depend on $x,\ t,\ \xi$

$$x = \frac{k^+ + k'^+}{p^+ + p'^+}, \qquad t = \Delta^2, \qquad \xi = \frac{p^+ - p'^+}{p^+ + p'^+},$$

(Average longitudinal momentum fraction of a parton) (Four momentum squared) (Skewness parameter)

- Define

$$P = \frac{p+p'}{2}, \quad \Delta = p' - p.$$

- The generalized parton distribution functions

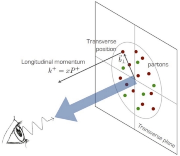
$$\Phi^{[\Gamma]}(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle p'|\overline{\psi}\left(-\frac{1}{2}z\right)\Gamma\psi\left(\frac{1}{2}z\right)|p\rangle\Big|_{z^+=0.\boldsymbol{z}^\perp=0},$$

Γ : Dirac matrix.

Γ	γ^+	$H^q,\ E^q$	unpol.
	$\gamma^+\gamma^5$	$\tilde{H}^q,\ \tilde{E}^q$	long. pol
	σ^{+i}	$H_T^q,\ E_T^q,\ \tilde{H}_T^q,\ \tilde{E}_T^q$	transv. pol.

- Impact parameter dependepnt parton distribution function

$$q(x,\boldsymbol{b}^\perp) = \int \frac{d^2\boldsymbol{\Delta}^\perp}{(2\pi)^2} e^{-i\boldsymbol{b}^\perp\cdot\boldsymbol{\Delta}^\perp} H^q\left(x,0,\Delta^{\perp 2}\right).$$



Gravitational form factors from GPDs

- The use of GPDs to address the physical content of GFFs was done by Ji in the context of the angular momentum decomposition of nucleons.

$$\int_{-1}^{1} dx\, x \left(H^a(x,\xi,t) + E^a(x,\xi,t)\right) = A^a(t) + B^a(t),$$

where $a = g,\ u,\ d,\ \ldots$ are type of partons. [X D. Ji (1997). *Phys. Rev. Lett.*, 78:610–613]

- The GPDs $H^q(x,\xi,t)$ and $E^q(x,\xi,t)$ give access to the quark GFFs as follows

$$\int_{-1}^{1} dx\, x H^q(x,\xi,t) = A^q(t) + \xi^2 D^q(t),$$

$$\int_{-1}^{1} dx\, x E^q(x,\xi,t) = B^q(t) - \xi^2 D^q(t).$$

[M V. Polyakov and P. Schweitzer (2018). *Int. J. Mod. Phys. A*, 33(26):1830025]

Light Front Dynamics and Light Front QCD

- According to Dirac there are practically three different parametrizations which are not accessible by a Lorentz transformation. [P A M. Dirac (1949). *Rev. Mod. Phys.*, 21:392–399]

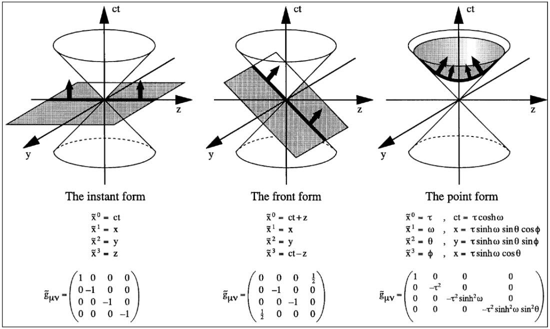


Figure: Three different forms of Hamiltonian dynamics. Image source [S J. Brodsky, H C. Pauli, and S S. Pinsky (1998). *Phys. Rept.*, 301:299–486].

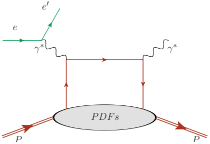
- Different parametrizations have their own Hamiltonian and thus according to Dirac, it is called different *forms of Hamiltonian dynamics*.

Light-front QCD

- According to the factorization theorem, for large momentum transfer, hadronic structure functions split into a hard, perturbative part and a soft, nonperturbative part that reflects the low-energy properties of the hadron's quarks and gluons. [J C. Collins, D E. Soper, and G F. Sterman (1989). *Adv. Ser. Direct. High Energy Phys.*, 5:1–91]

$$F_1(x) = \sum_q \frac{e_q^2}{2} f_{1q}(x),$$

$$F_2(x) = \sum_q e_q^2 \, x \, f_{1q}(x).$$



The $f_1(x)$ function has the explicit form

$$f_1(x) = \int \frac{d\xi^-}{8\pi} e^{-\frac{i}{2} x P^+ \xi^-} \langle P, S | \overline{\psi}^q \left(0, \xi^-, \mathbf{0}^\perp\right) \gamma^+ \psi^q(0) | P, S \rangle,$$

where, $\gamma^+ = \gamma^0 + \gamma^3$.

- Nonperturbative QCD dynamics, interpreted through parton distribution functions, is primarily governed by the noncollinear motion of low-energy quarks and gluons.

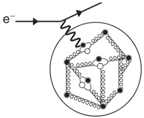


Image source: [M. Thomson (2013). Cambridge University Press, New York]

Light front QCD Hamiltonian

- Traditionally the light-front Hamiltonian is defined in a Lorentz invariant way as $H_{\mathrm{LF}} = P^{\mu}P_{\mu} = P^{-}P^{+} - \boldsymbol{P}^{\perp}$, where $P^{-} = P^{0} - P^{3}$ is the light-front time evolution operator, $P^{+} = P^{0} + P^{3}$ and $\boldsymbol{P}^{\perp}$ are longitudinal and transverse momentum respectively.

- We will denote P^{-} as H and call it the LFQCD Hamiltonian, which satisfies the eigenvalue equation

$$H\,|\psi\rangle = \frac{M^2 + P^{\perp 2}}{P^{+}}\,|\psi\rangle,$$

  and generates light-front time translation.

- H can be calculated from the QCD Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right) + \overline{\psi}\left(i\gamma_{\mu}D^{\mu} - m\right)\psi.$$

Bound State of Hadron

Equal time approach

- Solving relativistic bound state is equivalent to solving

$$H|\Psi\rangle = \sqrt{M^2 + P^2}\,|\Psi\rangle$$

- We can expand $|\Psi\rangle$ in terms of multi-parton Fock states as described by Tamm and Dancoff but has a complicated non-covariant structure.
  [I. Tamm (1945). *J. Phys. (USSR)*, 9:449] [S M. Dancoff (1950). *Phys. Rev.*, 78:382–385]
- Vacuum is complicated.
- The square root operator poses serious mathematical problems.
- The problem can be solved at the rest frame of the particle, however, the boosted wavefunction contains complex dynamical problems.

Light front approach

- One aims to solve the Hamiltonian eigenvalue problem

$$H|\Psi\rangle = \frac{M^2 + P^{\perp 2}}{P^+}\,|\Psi\rangle.$$

- We can also expand $|\Psi\rangle$ in terms of multi-parton Fock states in terms of light front wavefunction describes a fully relativistic system.
- Vacuum is less complicated.
- There is no square root operator in the equation.
- The boost operators are kinematic so the boosted wavefunction does not have complex dynamical problems.

- Any hadronic bound state $|\Psi\rangle$ of mass M must be an eigenstate of the light front Hamiltonian H_{LF} satisfying the eigenvalue equation $\left(H_{\mathrm{LF}} - M^2\right)|\Psi\rangle = 0$,

$$H\,|\Psi\rangle = \frac{M^2 + P^{\perp 2}}{P^+}\,|\Psi\rangle.$$

- $|\Psi\rangle$ can be expanded in terms of a complete set of functions $|n_i\rangle$

$$\int [dn_i]\,|n_i\rangle\langle n_i| = \mathbf{1},$$

where $[dn_i]$ is the phase-space differential.

- The projection of $|\Psi\rangle$ onto the basis states $|n_i\rangle$ are called *light front wavefunctions* (LFWF)

$$\Psi_n(n_i) \equiv \langle n_i|\Psi\rangle,$$

thus we can write

$$|\Psi\rangle = \sum_i \int [dn_i]\,|n_i\rangle\langle n_i|\Psi\rangle.$$

- We can construct the complete basis Fock states $|n_i\rangle$ by applying the free field creation operators to the vacuum $|0\rangle$

$$
\begin{aligned}
|q:\ k^+,\ \boldsymbol{k}^\perp,\ \lambda\rangle &= b^\dagger_\lambda(k)|0\rangle,\\
|qq:\ k_i^+,\ \boldsymbol{k}_i^\perp,\ \lambda_i\rangle &= b^\dagger_{\lambda_1}(k_1)b^\dagger_{\lambda_2}(k_2)|0\rangle,\\
|qqg:\ k_i^+,\ \boldsymbol{k}_i^\perp,\ \lambda_i\rangle &= b^\dagger_{\lambda_1}(k_1)b^\dagger_{\lambda_2}(k_2)a^\dagger_{\lambda_3}(k_3)|0\rangle,\\
|q\overline{q}:\ k_i^+,\ \boldsymbol{k}_i^\perp,\ \lambda_i\rangle &= b^\dagger_{\lambda_1}(k_1)d^\dagger_{\lambda_2}(k_2)|0\rangle,\\
&\cdots\ \text{so on.}
\end{aligned}
$$

  The operators here $b^\dagger(k),\ d^\dagger(k)$ create bare leptons and anti-leptons and $a^\dagger(k)$ create bare vector bosons with corresponding helicities λ_i.

- Each Fock state $|n_i\rangle = |n_i; k_i^+, \boldsymbol{k}_i^\perp, \lambda_i\rangle$ is an eigenstate of the operators P^+ and $\boldsymbol{P}^\perp$ with eigenvalues

$$
P^+\,|n_i\rangle = \left(\sum_i k_i^+\right)\,|n_i\rangle, \quad \boldsymbol{P}^\perp\,|n_i\rangle = \left(\sum_i \boldsymbol{k}_i^\perp\right)\,|n_i\rangle \quad \text{with } k_i^+ > 0.
$$

  The vacuum has the eigenvalues 0, $P^+\,|0\rangle = 0,\ \boldsymbol{P}^\perp\,|0\rangle = \boldsymbol{0}$.

- We can define boost invariant longitudinal momentum fractions x_i and relative transverse momentum $\boldsymbol{\kappa}_i^{\perp}$ as

$$x_i = \frac{k_i^+}{P^+}, \quad \text{with } 0 < x < 1,$$
$$\boldsymbol{k}_i^{\perp} = x_i \boldsymbol{P}^{\perp} + \boldsymbol{\kappa}_i^{\perp}$$

- In this notation particles in the Fock state have four-momentum

$$k_i^{\mu} \equiv \left(k_i^+, \boldsymbol{k}_i^{\perp}, k^-\right) = \left(x_i P^+, x_i \boldsymbol{P}^{\perp} + \boldsymbol{\kappa}_i^{\perp}, \frac{m_i^2 + \left(x_i \boldsymbol{P}^{\perp} + \boldsymbol{\kappa}_i^{\perp}\right)^2}{x_i P^+}\right),$$

and are on-shell i.e. $k^{\mu} k_{\mu} = m_i^2$.

- The value of x_i and $\boldsymbol{\kappa}_i^{\perp}$ are constrained by

$$\sum_i x_i = 1, \quad \sum_i \boldsymbol{\kappa}_i^{\perp} = \mathbf{0}.$$

- The eigenvalue equation

$$H\,|\Psi\rangle = \frac{M^2 + P^{\perp 2}}{P^+}\,|\Psi\rangle.$$

becomes

$$\sum_i \int [dn_i]\,\langle n_i; x_i, \boldsymbol{\kappa}_i^\perp, \lambda_i|\; H\,|n_i; x_i, \boldsymbol{\kappa}_i^\perp, \lambda_i\rangle\;\Psi_n\left(x_i, \boldsymbol{\kappa}_i^\perp, \lambda_i\right) = \frac{M^2 + P^{\perp 2}}{P^+}\;\Psi_n\left(x_i, \boldsymbol{\kappa}_i^\perp, \lambda_i\right),$$

for $i = 1, \ldots, \infty$.

- The phase-space differential as

$$[dn_i] = \delta\left(1 - \sum_i x_i\right)\delta\left(\sum_i \boldsymbol{\kappa}_i^\perp\right)\prod_i dx_i\; d^2\boldsymbol{\kappa}_i^\perp,$$

Discovery of partons

- The nucleon's response in DIS is described by structure functions dependent on the Lorentz invariants $p_i \cdot q$ and $Q^2 = -q^2$, where p_i^μ is the nucleon four momentum and q^μ is the four-momentum transfer.

- Bjorken scaling was observed in inclusive deep inelastic scattering (DIS) experiments at SLAC in 1969.

  [J D. Bjorken (1969). *Phys. Rev.*, 179:1547–1553]

- Bjorken scaling is the property that, in the high-energy limit $p_i \cdot q \to \infty$ and $Q^2 \to \infty$ with the ratio $x = \frac{Q^2}{2p_i \cdot q}$ fixed, the structure functions are functions of x, which satisfies $0 < x < 1$.