IMP, Huizhou,

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Few-body relativistic Light-Front wave functions: deuteron, He-3, nucleon, He-4

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• Light-front dynamics (LFD)

Dirac (1949)

State vector (wave function) is defined on the light-front plane

$$t + z = 0$$

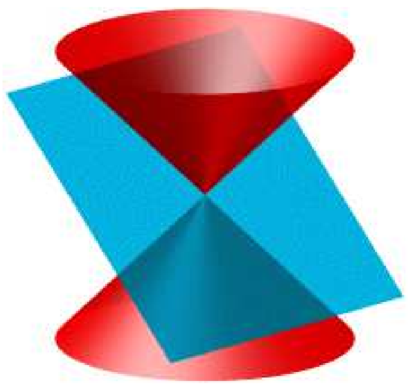
• Explicitly covariant LFD

V.A. Karmanov, JETP, **44** (1976) 201.

J. Carbonell, B. Desplanques, V.A. Karmanov, J.-F. Mathiot, Phys. Reports, **300** (1998) 215

State vector is defined on the light-front plane of general orientation:

$$\omega\cdot x = \omega_0 t - \vec{\omega}\cdot\vec{x}, \quad \omega = (\omega_0, \vec{\omega}), \quad \omega^2 = 0.$$



Particular case: $\omega = (1, 0, 0, -1)$ corresponds to the standard approach.

• History

J. Carbonell, V.A. Karmanov,

Relativistic deuteron wave function in the light-front dynamics,

Nucl. Phys. **A 581** (1995) 625.

Six spin components, each depends on two scalar variables:

$$\psi = \psi(\vec{k}_\perp, x)$$

Next step: three-fermion system.

V.A. Karmanov,

The nucleon wave function in light-front dynamics,

Nucl. Phys. **A 644** (1998) 165.

Sixteen spin components, each depends on five scalar variables:

$$\psi = \psi(\vec{k}_{1\perp}, \vec{k}_{2\perp}, \vec{k}_{3\perp}; x_1, x_2, x_3),\ \vec{k}_{1\perp} + \vec{k}_{2\perp} + \vec{k}_{3\perp} = 0,\ x_1 + x_2 + x_3 = 1$$

$$\to \psi = \psi(k_{1\perp}, k_{2\perp}, \vec{k}_{1\perp} \cdot \vec{k}_{2\perp}; x_1, x_2) \leftarrow$$ five scalar variables

This was too complicated for us!

Prof. Xingbo Zhao:

Don’t worry about the computer power.
We have computer facilities allowing to solve practically any interesting problem!

****

Another problem: labor force.

I hope to solve these very interesting and important problems together with you.
And then go ahead.

• Two-body wave function

$$\psi = \psi(k_1, k_2, p, \omega\tau), \quad k_1 + k_2 = p + \omega\tau$$

$$k_1^2 = k_2^2 = m^2,\ p^2 = M^2,\ (\omega\tau)^2 = 0.$$

On mass shell, but off-energy-shell!

If $\omega = (1, 0, 0, -1)$, $\vec{\omega}_\perp = 0, \omega_+ = \omega_0 + \omega_z = 0, \omega_- = 2$,

we restore the ordinary version of LFD:

$$\vec{k}_{1\perp} + \vec{k}_{2\perp} = \vec{p}_\perp,\ k_{1+} + k_{2+} = p_+,$$

but $k_{1-} + k_{2-} - p_- = 2\tau \neq 0.$

C.m. variables: $\vec{k}_1 + \vec{k}_2 = \vec{p} + \vec{\omega}\tau = 0$

$$\rightarrow\ \vec{k} = \vec{k}_1 = -\vec{k}_2,\ \vec{n} = \vec{\omega}/|\vec{\omega}|, \quad \psi = \psi(\vec{k}, \vec{n}) \equiv \psi(k_\perp, x)$$

since

$$\vec{k}_\perp = \vec{k} - (\vec{n}\vec{k})\vec{n}, \quad \vec{k}_\perp^2 = \vec{k}^2 - (\vec{n}\vec{k})^2, \quad x = \frac{1}{2}\left(1 - \frac{\vec{n}\vec{k}}{\sqrt{\vec{k}^2 + m^2}}\right)$$

• Spinless equation

$$\left(\frac{\vec{k}_\perp^2 + m^2}{x(1-x)} - M^2\right)\psi(\vec{k}_\perp, x)$$

$$= \quad -\frac{m^2}{2\pi^3}\int \psi(\vec{k}'_\perp, x')V(\vec{k}'_\perp, x'; \vec{k}_\perp, x, M^2)\frac{d^3k'_\perp dx'}{2x'(1-x')}$$

Kernel (one-boson exchange):

$$V(\vec{k}'_\perp, x'; \vec{k}_\perp, x, M^2) =$$

$$g^2\left[\mu^2 + \frac{x'}{x}\left(1 - \frac{x}{x'}\right)^2 m^2 + \frac{x'}{x}\left(\vec{k}_\perp - \frac{x}{x'}\vec{k}'_\perp\right)^2\right.$$
$$\left. + (x'-x)\left(\frac{m^2 + \vec{k}_\perp^2}{x(1-x)} - M^2\right) + \mu^2\right]^{-1}, \quad x' > x$$

• Non-relativistic deuteron WF

Bound np system, $J^\pi = 1^+$.

$$\psi^M_{NR}(\vec{k}) = w^\dagger_{\sigma_1}\left[\frac{1}{\sqrt{4\pi}} u_S(k) C^{1M}_{\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2} + u_D(k) C^{1\sigma}_{\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2} C^{1M}_{1\sigma 2m} Y_{2m}\left(\frac{\vec{k}}{k}\right)\right] w^\dagger_{\sigma_2}$$

Two scalar functions (spin components):

$u_S(k)$, S-wave ($L=0$): $J = (\frac{1}{2}+\frac{1}{2}=1)+0 \Rightarrow 1$

$u_D(k)$, D-wave ($L=2$): $J = (\frac{1}{2}+\frac{1}{2}=1)+2 \Rightarrow 1$

Or, the same via Pauli matrices:

$$\vec{\psi}_{NR}(\vec{k}\,) = w^\dagger_{\sigma_1}\left[u_S(k)\frac{1}{\sqrt{2}}\vec{\sigma} - u_D(k)\frac{1}{2}\left(\frac{3\vec{k}(\vec{k}\cdot\vec{\sigma})}{\vec{k}^{\,2}} - \vec{\sigma}\right)\right] w^\dagger_{\sigma_2}\ .$$

• Relativistic deuteron LFWF

We have extra vector parameter $\vec{n}$.

$$\begin{aligned}
\vec{\psi}(\vec{k},\vec{n}) &= w^{\dagger}_{\sigma_1}\Bigg[f_1\frac{1}{\sqrt{2}}\vec{\sigma}+f_2\frac{1}{2}\Big(\frac{3\vec{k}(\vec{k}\cdot\vec{\sigma})}{\vec{k}^2}-\vec{\sigma}\Big)\\
&+ f_3\frac{1}{2}(3\vec{n}(\vec{n}\cdot\vec{\sigma})-\vec{\sigma})\\
&+ f_4\frac{1}{2k}(3\vec{k}(\vec{n}\cdot\vec{\sigma})+3\vec{n}(\vec{k}\cdot\vec{\sigma})-2(\vec{k}\cdot\vec{n})\vec{\sigma})\\
&+ f_5\sqrt{\frac{3}{2}}\frac{i}{k}[\vec{k}\times\vec{n}]\\
&+ f_6\frac{\sqrt{3}}{2k}[[\vec{k}\times\vec{n}]\times\vec{\sigma}]\Bigg]\sigma_y w^{\dagger}_{\sigma_2}
\end{aligned}$$

Six spin components $f_1 - f_6$ instead of two!

• Another (4D) representation

$$
\begin{aligned}
\Phi^{\lambda}_{\sigma_2\sigma_1}(k_1,k_2,p,\omega\tau) &= \sqrt{m}\, e_\mu(p,\lambda)\ \bar{u}^{\sigma_2}(k_2) \\
\times & \left[\varphi_1 \frac{(k_1-k_2)^\mu}{2m^2} + \varphi_2 \frac{1}{m}\gamma^\mu + \varphi_3 \frac{\omega^\mu}{\omega\cdot p} + \varphi_4 \frac{(k_1-k_2)^\mu \hat{\omega}}{2m\omega\cdot p}\right. \\
- & \left.\varphi_5 \frac{i}{m^2\omega\cdot p}\gamma_5 \epsilon^{\mu\nu\rho\gamma} k_{1\nu} k_{2\rho}\omega_\gamma + \varphi_6 \frac{m\omega^\mu \hat{\omega}}{(\omega\cdot p)^2}\right]\ U_c \bar{u}^{\sigma_1}(k_1)\ ,
\end{aligned}
$$

where $e_\mu(p,\lambda)$ is the deuteron polarization vector.

At $\vec{k}_1 + \vec{k}_2 = 0$ this wave function turns into $\vec{\psi}(\vec{k},\vec{n})$.

One can use both representations, depending on convenience.

• Why six components?

$$\Phi^{\lambda}_{\sigma_2\sigma_1}$$

$$\sigma_1 = \pm\frac{1}{2},\ \sigma_2 = \pm\frac{1}{2},\ \lambda = -1, 0, 1$$

$$\Rightarrow 2 \times 2 \times 3 = 12$$

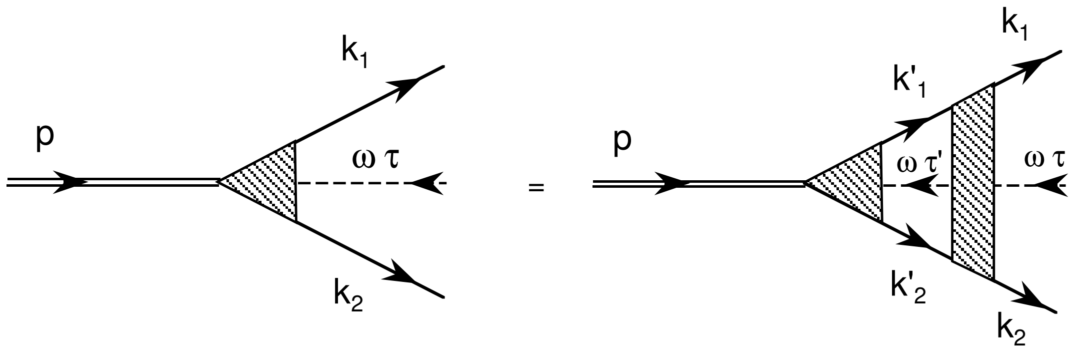
parity conservation $\Rightarrow 12/2 = 6$

Important:

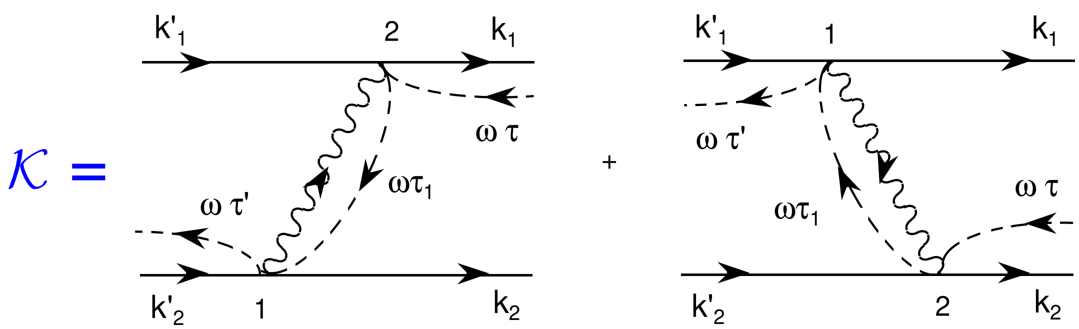
Parity conservation does not reduce the number of components for the non-relativistic 4-body WF and for 3-body LFWF!

(to be explained later)

• Equation for two-body LFWF



• Kernel (OBE)



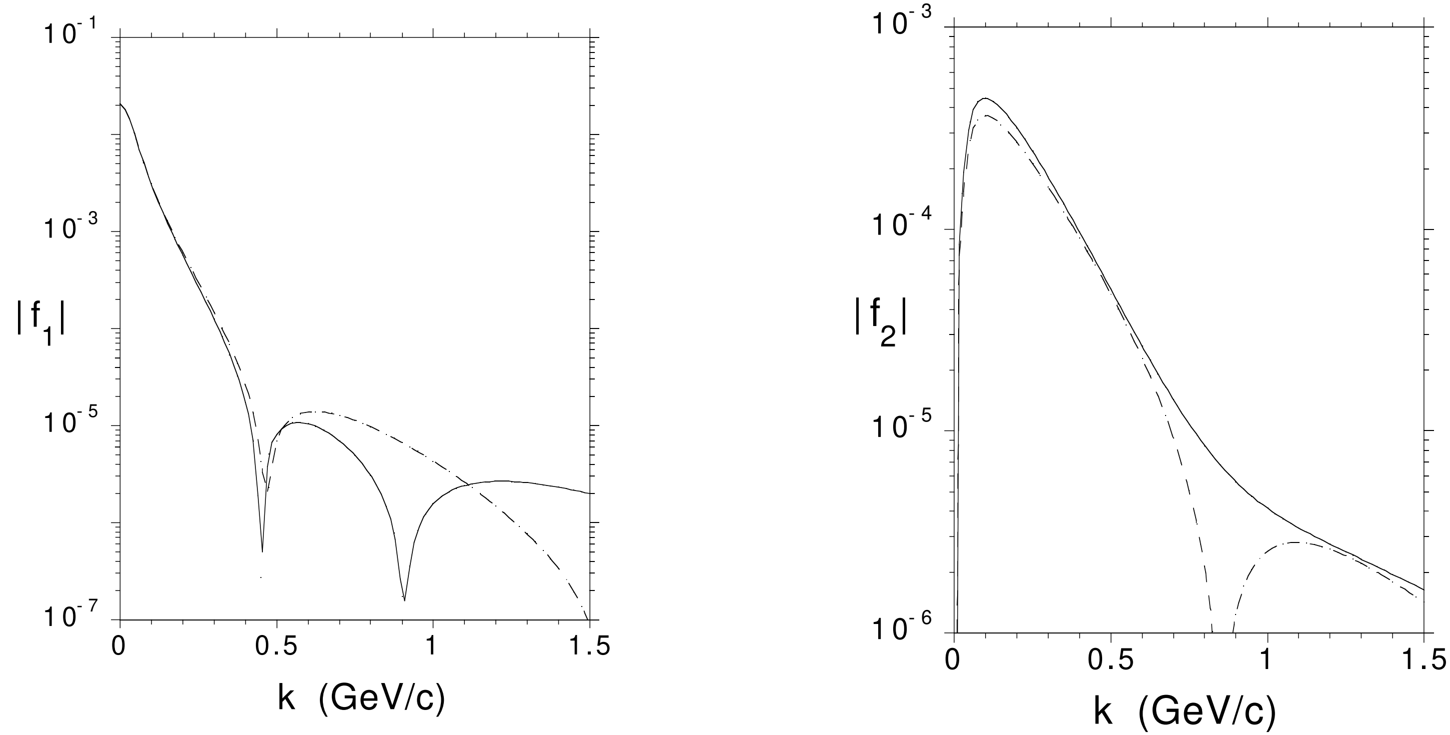
= sum over seven meson exchanges

• The meson's parameters

Table 1: Parameters of the exchanged mesons (Bonn potential)

	J^π	T	μ (MeV)	$g^2/(4\pi)$ $[f/g]$	Λ (GeV)	n
π	0^-	1	138.03	14.6	1.3	1
η	0^-	0	548.8	5.0	1.5	1
δ	0^+	1	983	1.1075	2	1
σ_0	0^+		720	16.9822	2	1
σ_1	0^+		550	8.2797	2	1
ω	1^-	0	782.6	20.0 [0.0]	1.5	1
ρ	1^-	1	769	0.81 [6.1]	2	2

• Numerical results for f_1, f_2

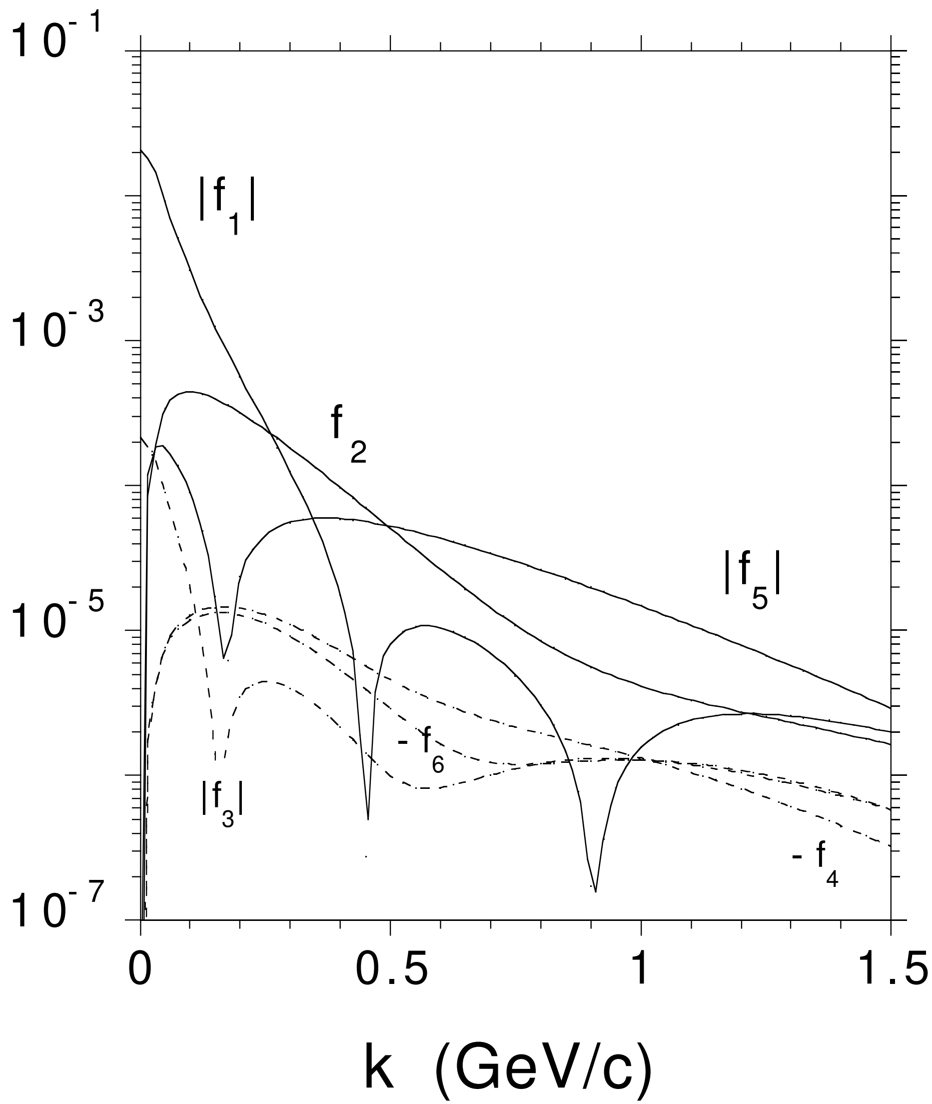


(a) (b)

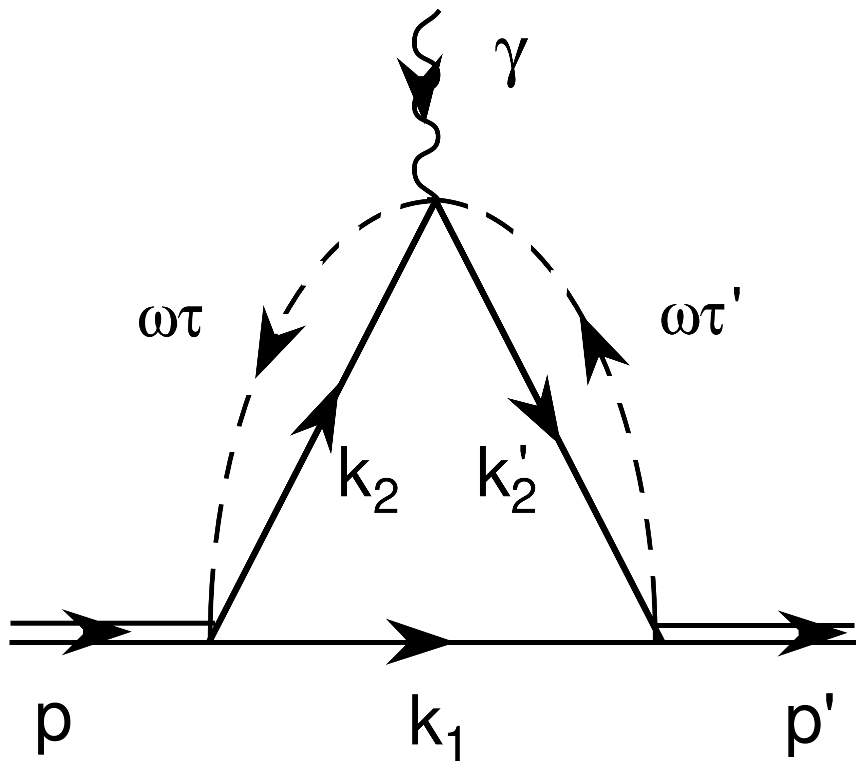
Dashed – non-relativistic
Solid line – relativistic (LFD)

• Numerical results for all f_{1-6}

(J. Carbonell & V.A. Karmanov)



• E.M. form factors



• Deuteron electromagnetic form factors

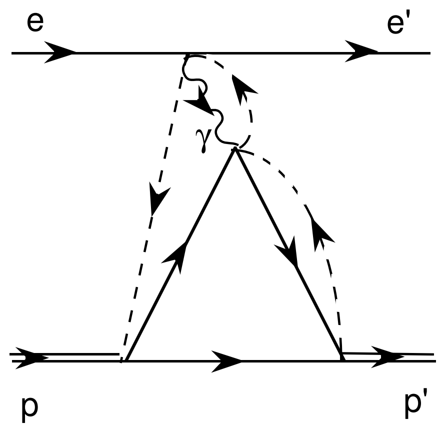
$$\langle\lambda'|J_\rho|\lambda\rangle = e^{*\lambda'}_\mu(p')\left\{P_\rho\left[\mathcal{F}_1(q^2)g^{\mu\nu} + \mathcal{F}_2(q^2)\frac{q^\mu q^\nu}{2M^2}\right]\right.$$
$$\left. + \mathcal{G}_1(q^2)(g^\mu_\rho q^\nu - g^\nu_\rho q^\mu)\right\} e^\lambda_\nu(p)$$

Charge, magnetic and quadrupole form factors

$$F_C = -\mathcal{F}_1 - \frac{2\eta}{3}[\mathcal{F}_1 + \mathcal{G}_1 - \mathcal{F}_2(1+\eta)] ,$$
$$F_M = \mathcal{G}_1 ,$$
$$F_Q = -\mathcal{F}_1 - \mathcal{G}_1 + \mathcal{F}_2(1+\eta),$$ where $\eta = Q^2/4M^2$.

• ed cross section

ed scattering amplitude:



$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \left(A(q^2) + \tan^2 \frac{1}{2}\theta \; B(q^2) \right)$$

$$\begin{aligned} A(q^2) &= F_C^2(q^2) + \frac{8}{9}\eta^2 F_Q^2(q^2) + \frac{2}{3}\eta F_M^2(q^2) \, , \\ B(q^2) &= \frac{4}{3}\eta(1+\eta) F_M^2(q^2) \, . \end{aligned}$$

• Polarization observable t_{20}

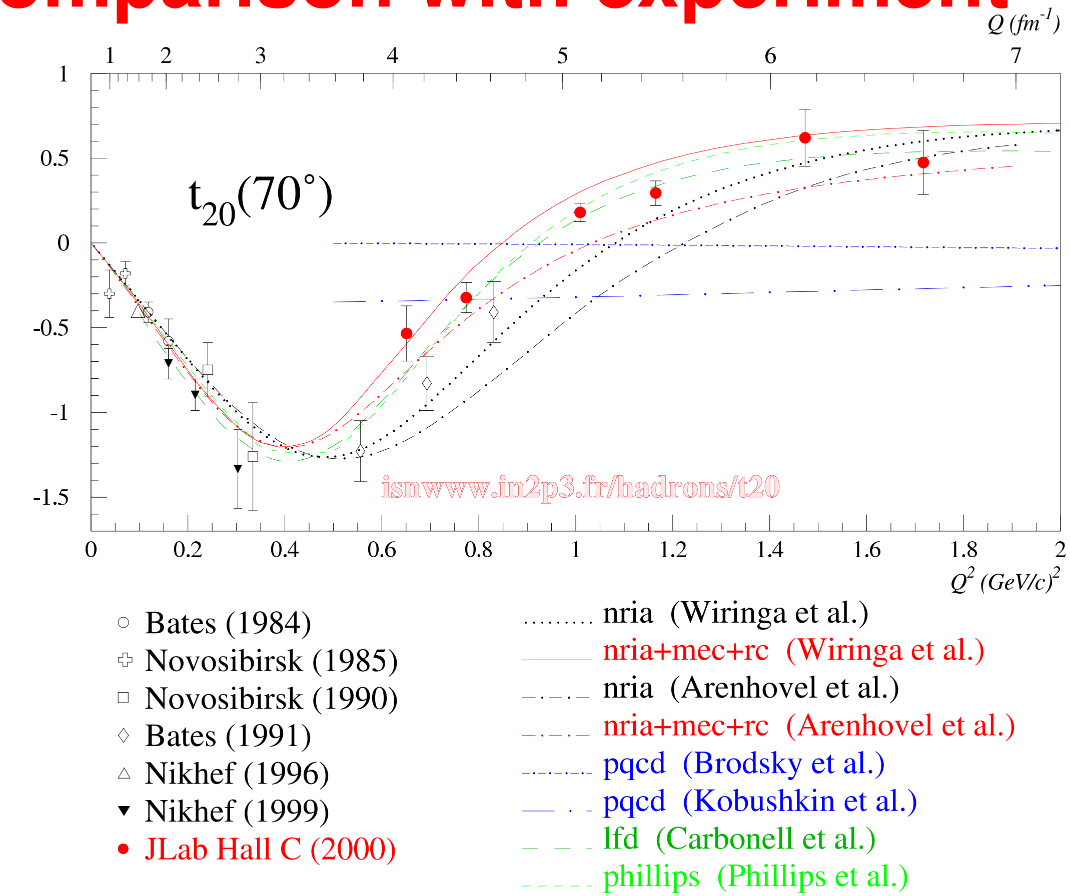
$$t_{20} = \frac{\left[\left(\frac{d\sigma}{d\Omega}\right)_+ - 2\left(\frac{d\sigma}{d\Omega}\right)_0 + \left(\frac{d\sigma}{d\Omega}\right)_-\right]}{\left[\left(\frac{d\sigma}{d\Omega}\right)_+ + \left(\frac{d\sigma}{d\Omega}\right)_0 + \left(\frac{d\sigma}{d\Omega}\right)_-\right]}$$

$$t_{20}\left(A(q^2) + \tan^2\frac{1}{2}\theta\ B(q^2)\right) =$$

$$-\frac{1}{\sqrt{2}}\left[\frac{8}{3}\eta F_C F_Q + \frac{8}{9}\eta^2 F_Q^2 + \frac{1}{3}\eta\left(1 + 2(1+\eta)\tan^2\frac{1}{2}\theta\right)F_M^2\right].$$

• t_{20} prediction

Comparison with experiment



End of lecture 1

• Lecture 2.

Reminder of the lecture 1. Deuteron LF wave function.

Wave function $\Phi^{\lambda}_{\sigma_2\sigma_1}$ of a system with total momentum $J=1$, its projections λ, made of two fermions (spins 1/2, projections $\sigma_{1,2}=\pm\frac{1}{2}$), is a matrix containing $2\times 2\times 3=12$ elements. Due to parity conservation, only half of them is independent: $\Rightarrow$12/2=6.

In principle, to find $\Phi^{\lambda}_{\sigma_2\sigma_1}$, we have to find (from equation) all these six independent matrix elements (as functions of the particle momenta).

Our strategy: we represent $\Phi^{\lambda}_{\sigma_2\sigma_1}$ as a sum of six known matrices with unknown scalar coefficients φ_{1-6} (called spin components)

$$
\begin{aligned}
\Phi^{\lambda}_{\sigma_2\sigma_1}(k_1,k_2,p,\omega\tau) &= \mathcal{O}^{\lambda}_{1\sigma_2\sigma_1}\varphi_1+\mathcal{O}^{\lambda}_{2\sigma_2\sigma_1}\varphi_2+\mathcal{O}^{\lambda}_{3\sigma_2\sigma_1}\varphi_3\\
&+ \mathcal{O}^{\lambda}_{4\sigma_2\sigma_1}\varphi_4+\mathcal{O}^{\lambda}_{5\sigma_2\sigma_1}\varphi_5+\mathcal{O}^{\lambda}_{6\sigma_2\sigma_1}\varphi_6
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{\lambda}_{\sigma_2\sigma_1}(k_1,k_2,p,\omega\tau) &= \sqrt{m}\, e_\mu(p,\lambda)\ \bar{u}^{\sigma_2}(k_2) \\
\times & \left[\varphi_1 \frac{(k_1-k_2)^\mu}{2m^2} + \varphi_2 \frac{1}{m}\gamma^\mu + \varphi_3 \frac{\omega^\mu}{\omega\cdot p} + \varphi_4 \frac{(k_1-k_2)^\mu \hat{\omega}}{2m\omega\cdot p} \right. \\
- & \left. \varphi_5 \frac{i}{m^2\omega\cdot p}\gamma_5 \epsilon^{\mu\nu\rho\gamma} k_{1\nu} k_{2\rho} \omega_\gamma + \varphi_6 \frac{m\omega^\mu \hat{\omega}}{(\omega\cdot p)^2}\right]\ U_c \bar{u}^{\sigma_1}(k_1)\ ,
\end{aligned}
$$

$$
e_\mu(p,\lambda)\ \bar{u}^{\sigma_2}(k_2)\mathcal{M}_1^\mu U_c \bar{u}^{\sigma_1}(k_1)\varphi_1 +
$$

That is:

$$
\begin{aligned}
\mathcal{O}^{\lambda}_{1\sigma_2\sigma_1} &= \sqrt{m}\, e_\mu(p,\lambda)\ \bar{u}^{\sigma_2}(k_2)\frac{(k_1-k_2)^\mu}{2m^2}\ U_c \bar{u}^{\sigma_1}(k_1) \\
\mathcal{O}^{\lambda}_{2\sigma_2\sigma_1} &= \ldots \\
\mathcal{O}^{\lambda}_{6\sigma_2\sigma_1} &= \sqrt{m}\, e_\mu(p,\lambda)\ \bar{u}^{\sigma_2}(k_2)\frac{m\omega^\mu\hat{\omega}}{(\omega\cdot p)^2}\ U_c \bar{u}^{\sigma_1}(k_1)
\end{aligned}
$$

• C.m. frame

$$\begin{aligned}
\vec{\psi}(\vec{k},\vec{n}) &= w^\dagger_{\sigma_2}\Bigg[f_1\frac{1}{\sqrt{2}}\vec{\sigma} + f_2\frac{1}{2}\Big(\frac{3\vec{k}(\vec{k}\cdot\vec{\sigma})}{\vec{k}^2} - \vec{\sigma}\Big) \\
&+ f_3\frac{1}{2}(3\vec{n}(\vec{n}\cdot\vec{\sigma}) - \vec{\sigma}) \\
&+ f_4\frac{1}{2k}(3\vec{k}(\vec{n}\cdot\vec{\sigma}) + 3\vec{n}(\vec{k}\cdot\vec{\sigma}) - 2(\vec{k}\cdot\vec{n})\vec{\sigma}) \\
&+ f_5\sqrt{\frac{3}{2}}\frac{i}{k}[\vec{k}\times\vec{n}] \\
&+ f_6\frac{\sqrt{3}}{2k}[[\vec{k}\times\vec{n}]\times\vec{\sigma}]\Bigg]\sigma_y w^\dagger_{\sigma_1}
\end{aligned}$$

$$\vec{O}_{1\sigma_2\sigma_1} = w^\dagger_{\sigma_2}\frac{1}{\sqrt{2}}\vec{\sigma}\sigma_y w^\dagger_{\sigma_1},\ \vec{O}_{2\sigma_2\sigma_1} = \ldots,\ \vec{O}_{6\sigma_2\sigma_1} = w^\dagger_{\sigma_2}\frac{\sqrt{3}}{2k}[[\vec{k}\times\vec{n}]\times\vec{\sigma}]\sigma_y w^\dagger_{\sigma_1}$$

• Trivial example

Take matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

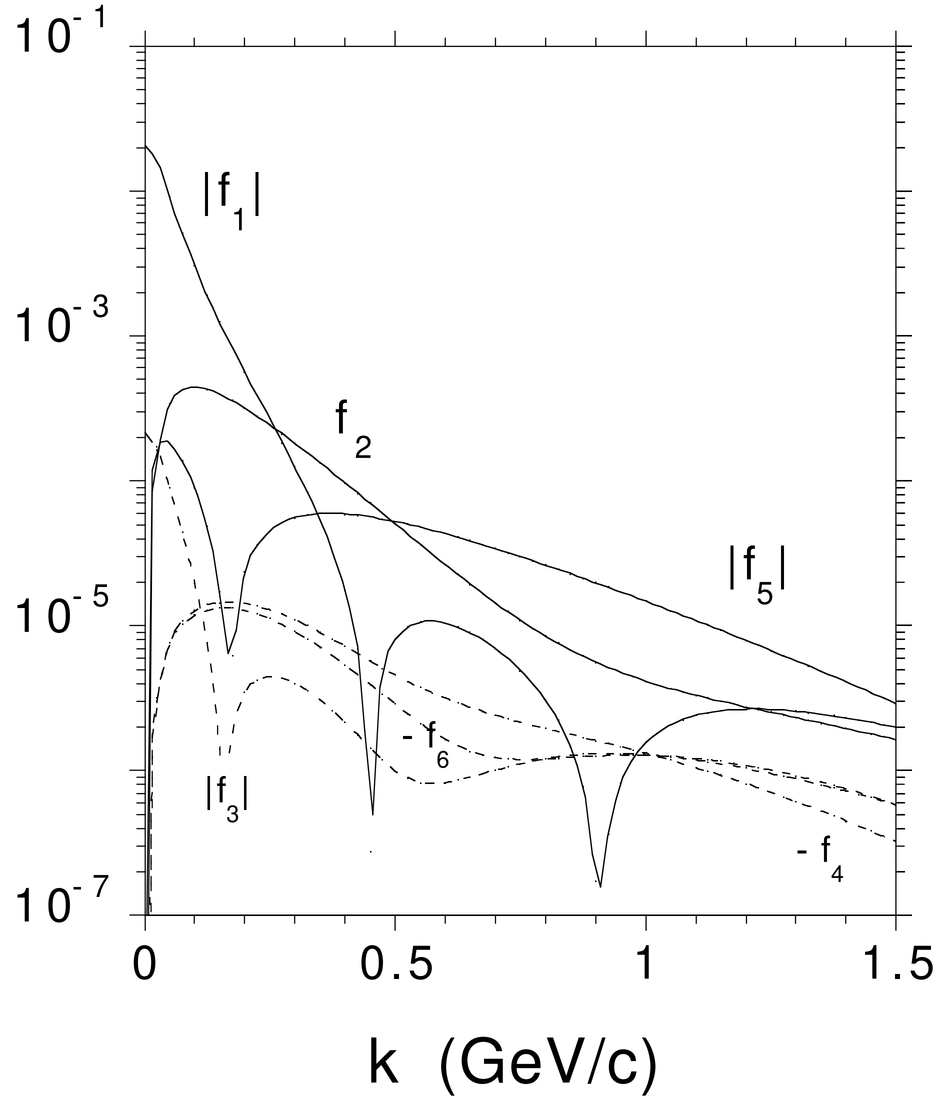
Decompose it as

$$M = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathcal{O}_1} + b \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathcal{O}_2} + c \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\mathcal{O}_3} + d \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathcal{O}_4}$$

The basis matrices $\mathcal{O}_{1-4}$ are known.
The matrix M is determined by the coefficients a, b, c, d,
similarly to the deuteron wave function.

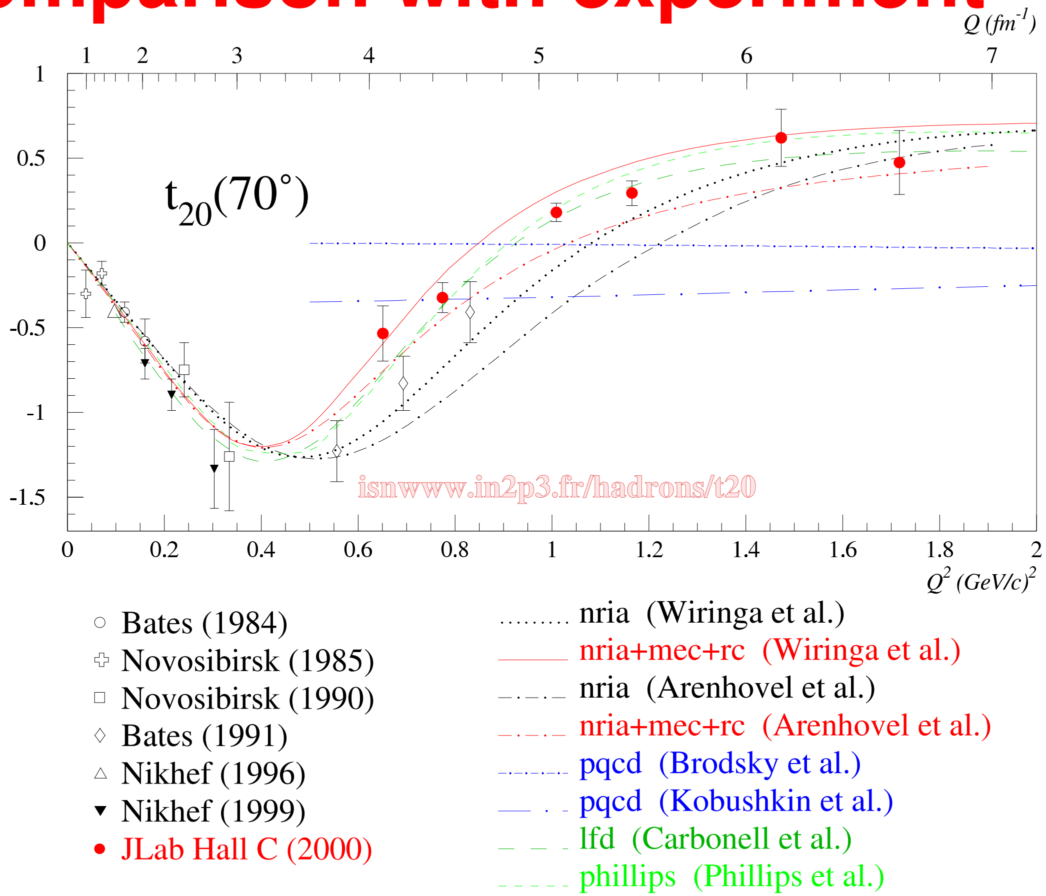
• Numerical results for all f_{1-6}

(J. Carbonell & V.A. Karmanov)



• t_{20} prediction

Comparison with experiment



End of reminder of lecture 1

• Three-body system

Faddeev components

Three-body Schrödinger equation

$$\left[-\frac{1}{2m}\Delta + V_1 + V_2 + V_3\right]\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3) = E\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)$$

Represent $\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3)$ as:

$$\Psi = \Psi_1(\vec{x}_1,\vec{y}_1) + \Psi_2(\vec{x}_2,\vec{y}_2) + \Psi_3(\vec{x}_3,\vec{y}_3).$$

where $\vec{x}_i,\vec{y}_i$ (i=1,2,3) are three sets of Jacobi coordinates

$$\vec{x}_1 = \vec{r}_2 - \vec{r}_3, \quad \vec{y}_1 = \frac{2}{\sqrt{3}}\left(\frac{\vec{r}_2+\vec{r}_3}{2} - \vec{r}_1\right),$$

and other pairs of coordinates are obtained by the cyclic permutation. Then the three-body Schrödinger equation is rewritten as the system of equations for the Faddeev components:

$$\left[k^2 + \Delta_{\vec{x}_1} + \Delta_{\vec{y}_1} - \frac{m}{\hbar^2} V_1(\vec{x}_1)\right] \Psi_1(\vec{x}_1, \vec{y}_1) =$$
$$\frac{m}{\hbar^2} V_1(\vec{x}_1)[\Psi_2(\vec{x}_2, \vec{y}_2) + \Psi_3(\vec{x}_3, \vec{y}_3)],$$
$$\left[k^2 + \Delta_{\vec{x}_2} + \Delta_{\vec{y}_2} - \frac{m}{\hbar^2} V_2(\vec{x}_2)\right] \Psi_2(\vec{x}_2, \vec{y}_2) =$$
$$\frac{m}{\hbar^2} V_2(\vec{x}_2)[\Psi_3(\vec{x}_3, \vec{y}_3) + \Psi_1(\vec{x}_1, \vec{y}_1)],$$
$$\left[k^2 + \Delta_{\vec{x}_3} + \Delta_{\vec{y}_3} - \frac{m}{\hbar^2} V_3(\vec{x}_3)\right] \Psi_3(\vec{x}_3, \vec{y}_3) =$$
$$\frac{m}{\hbar^2} V_3(\vec{x}_3)[\Psi_1(\vec{x}_1, \vec{y}_1) + \Psi_2(\vec{x}_2, \vec{y}_2)]. \quad (1)$$

where $k^2 = mE/\hbar^2$.

Sum of these equations gives initial three-body Schrödinger equation for Ψ.

• ^{3}He WF (ppn)

Permutations

$$\Psi(1,2,3) = \Phi_{12}(1,2,3) + \Phi_{12}(2,3,1) + \Phi_{12}(3,1,2),$$

$\Phi(1,2,3) = -\Phi_{12}(2,1,3)$ is Faddeev components. It is antisymmetric relative to permutation of the first pair 12 only.

Then $\Psi(1,2,3)$ is antisymmetric relative to permutation of the any pair.

For example:

$$\begin{aligned}
\Psi(1,2,3) &= \Phi_{12}(1,2,3) + \Phi_{12}(2,3,1) + \Phi_{12}(3,1,2) \\
\Psi(1,3,2) &= \Phi_{12}(1,3,2) + \Phi_{12}(3,2,1) + \Phi_{12}(2,1,3) \\
&= -\Phi_{12}(3,1,2) - \Phi_{12}(2,3,1) - \Phi_{12}(1,2,3) \\
&= -\Psi(1,2,3)
\end{aligned}$$

• Non-relativistic ^{3}He WF

$$J^\pi = \frac{1}{2}^+$$

Table 2: Spins, angular momenta and isospins forming the non-relativistic ^{3}He wave function.

spins-angular momenta								isospins		
n	S_{12}	L_{12}	J_{12}	s_3	l_3	j_3	J	T_{12}	t_3	T
1	0	0	0	1/2	0	1/2	1/2	1	1/2	1/2
2	1	0	1	1/2	0	1/2	1/2	0	1/2	1/2
3	1	2	1	1/2	0	1/2	1/2	0	1/2	1/2
4	1	0	1	1/2	2	3/2	1/2	0	1/2	1/2
5	1	2	1	1/2	2	3/2	1/2	0	1/2	1/2

• Non-relativistic ^{3}He WF

Spin basis

$$
\begin{aligned}
\chi_1 &= C^{00}_{\frac{1}{2}\sigma_1\,\frac{1}{2}\sigma_2} C^{\frac{1}{2}\sigma}_{00\,\frac{1}{2}\sigma_3} Y_{00} Y_{00} \\
\chi_2 &= \sum C^{1\sigma_{12}}_{\frac{1}{2}\sigma_1\,\frac{1}{2}\sigma_2} C^{\frac{1}{2}\sigma}_{1\sigma_{12}\,\frac{1}{2}\sigma_3} Y_{00} Y_{00} \\
\chi_3 &= \sum C^{1\sigma_{12}}_{\frac{1}{2}\sigma_1\,\frac{1}{2}\sigma_2} C^{1\sigma'}_{2m\,1\sigma_{12}} Y_{2m}\left(\frac{\vec{p}}{|\vec{p}|}\right) C^{\frac{1}{2}\sigma}_{1\sigma'\,\frac{1}{2}\sigma_3} Y_{00} \\
\chi_4 &= \sum C^{1\sigma_{12}}_{\frac{1}{2}\sigma_1\,\frac{1}{2}\sigma_2} C^{\frac{3}{2}\sigma'}_{2m\,\frac{1}{2}\sigma_3} Y_{2m}\left(\frac{\vec{q}}{|\vec{q}|}\right) C^{\frac{1}{2}\sigma}_{1\sigma_{12}\,\frac{3}{2}\sigma'} Y_{00} \\
\chi_5 &= \sum C^{1\sigma_{12}}_{\frac{1}{2}\sigma_1\,\frac{1}{2}\sigma_2} C^{1\sigma'}_{2m\,1\sigma_{12}} Y_{2m}\left(\frac{\vec{p}}{|\vec{p}|}\right) \\
&\qquad \times C^{\frac{3}{2}\sigma''}_{2m'\,\frac{1}{2}\sigma_3} Y_{2m'}\left(\frac{\vec{q}}{|\vec{q}|}\right) C^{\frac{1}{2}\sigma}_{1\sigma'\,\frac{3}{2}\sigma''},
\end{aligned}
$$

$$
\vec{p} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2), \quad \vec{q} = \frac{1}{3}\left(\frac{\vec{k}_1 + \vec{k}_2}{2} - \vec{k}_3\right).
$$

Isospin basis

$$\xi^{(0)}_{\tau_1\tau_2\tau_3} = C^{00}_{\frac{1}{2}\tau_1\ \frac{1}{2}\tau_2} C^{\frac{1}{2}\tau}_{00\frac{1}{2}\tau_3}$$

$$\xi^{(1)}_{\tau_1\tau_2\tau_3} = C^{1\tau_{12}}_{\frac{1}{2}\tau_1\ \frac{1}{2}\tau_2} C^{\frac{1}{2}\tau}_{1\tau_{12}\frac{1}{2}\tau_3}$$

Non-relativistic WF:

$$\begin{aligned}\Phi_{12} &= \psi_1(p,q)\chi_1\xi^{(1)}_{\tau_1\tau_2\tau_3} \\ &+ [\psi_2(p,q)\chi_2 + \psi_3(p,q)\chi_3 + \psi_4(p,q)\chi_4 + \psi_5(p,q)\chi_5]\xi^{(0)}_{\tau_1\tau_2\tau_3}\end{aligned}$$

• How many components?

$$\Phi_{12}(1,2,3) = \Phi_{12,\sigma}^{\sigma_1\sigma_2\sigma_3}(1,2,3)$$

$$\sigma_1, \sigma_2, \sigma_3, \sigma = \pm\frac{1}{2} \quad \rightarrow \quad 2\times 2\times 2\times 2 = 16.$$

Parity conservation **does not** reduce the number of components for the non-relativistic 4-body WF and for 3-body LFWF!

We can construct the pseudoscalar

$$C_{ps} = \frac{e^{\mu\nu\rho\gamma} k_{1\mu} k_{2\nu} p_\rho \omega_\gamma}{|e^{\mu\nu\rho\gamma} k_{1\mu} k_{2\nu} p_\rho \omega_\gamma|}$$

Parity conservation splits, as usual, 16 basis wave function in two groups: with positive and negative parity. However, multiplying, say, the functions with negative parities by C_{ps}, we restore the positive parity.

We get the total number: 16

This happens in the 3-body relativistic case, when the wave function depends on the orientation of the LF plane.

In two-body relativistic (and non-relativistic case) it is impossible ($C_{ps} = 0$)

In three-body non-relativistic case it is also impossible: $C_{ps} = e^{\mu\nu\rho\gamma} k_{1\mu} k_{2\nu} k_{3\rho} p_\gamma = 0$ (since $p = k_1 + k_2 + k_3$).

In 4-body non-relativistic case it is possible (known long ago).

The problems to be solved:

1. To construct 16 basis functions and to decompose jn this basis the LFWF $\Phi_{12}(1,2,3)$ (Faddeev component).
2. To derive the system of equations for the coefficients these decomposition.
3. To solve this system and find in this way the ^{3}He LFWF.
4. (Next step) Using this solution, to calculate the ^{3}He em FF's.

• Orthogonal spin basis

$$
\begin{aligned}
s_1 &= \left(2x_3 - (m + x_3 M)\frac{\hat{\omega}}{\omega\cdot p}\right), \\
s_2 &= \frac{m}{\omega\cdot p}\hat{\omega}, \\
s_3 &= i\left(2x_3 - (m - x_3 M)\frac{\hat{\omega}}{\omega\cdot p}\right)\gamma_5, \\
s_4 &= \frac{im}{\omega\cdot p}\hat{\omega}\gamma_5.
\end{aligned}
$$

$$
\hat{\omega} = \omega_\mu \gamma^\mu, \;\; x_3 = \frac{\omega\cdot k_3}{\omega\cdot p}.
$$

Define (nucleon No. 3 and ^{3}He):

$$
\begin{aligned}
S^j_{\sigma\sigma_3} &= N^S_j \bar{u}_\sigma(k_3) s_j u_{\sigma_3}(p), \\
\bar{S}^j_{\sigma\sigma_3} &= N^S_j \bar{u}_\sigma(p) \bar{s}_j u_{\sigma_3}(k_3), \quad \bar{s}_j = \gamma_0 s_j^\dagger \gamma_0.
\end{aligned}
$$

Orthogonality: $\frac{1}{2}\sum_{\sigma_3\sigma} \bar{S}^j_{\sigma\sigma_3} S^j_{\sigma\sigma_3} = \delta_{jj'}$.

• Another set

$$t_1 = i\left(2\frac{x_1x_2}{x_1+x_2} - \frac{m\hat{\omega}}{\omega{\cdot}p}\right)\gamma_5,$$

$$t_2 = i\frac{m\hat{\omega}}{\omega{\cdot}p}\gamma_5,$$

$$t_3 = \left(2\frac{x_1x_2}{(x_1+x_2)} + \frac{(x_1-x_2)}{(x_1+x_2)}\frac{m\hat{\omega}}{\omega{\cdot}p}\right)$$

$$t_4 = \frac{m\hat{\omega}}{\omega{\cdot}p}, \quad x_{1,2} = \frac{\omega{\cdot}k_{1,2}}{\omega{\cdot}p}.$$

Define (nucleons No. 1 and 2):

$$T^i_{\sigma_1\sigma_2} = N_i^T \bar{u}_{\sigma_1}(k_1) t_i U_c \bar{u}_{\sigma_2}(k_2), \quad U_c = \gamma_2\gamma_0,$$

$$\bar{T}^i_{\sigma_2\sigma_1} = N_i^T \bar{u}_{\sigma_2}(k_2) t_i^\dagger U_c \bar{u}_{\sigma_1}(k_1).$$

Orthogonality: $\sum_{\sigma_1\sigma_2} \bar{T}^i_{\sigma_2\sigma_1} T^{i'}_{\sigma_1\sigma_2} = \delta_{ii'}.$

• 16 terms basis V_{ij}

$$
\begin{aligned}
V_{11} &= T_1 \otimes S_1, \quad V_{12} = T_1 \otimes S_2, \\
V_{21} &= T_2 \otimes S_1, \quad V_{22} = T_2 \otimes S_2, \\
V_{33} &= T_3 \otimes S_3, \quad V_{34} = T_3 \otimes S_4, \\
V_{43} &= T_4 \otimes S_3, \quad V_{44} = T_4 \otimes S_4, \\
V_{13} &= T_1 \otimes S_3 C_{ps}, \quad V_{14} = T_1 \otimes S_4 C_{ps}, \\
V_{23} &= T_2 \otimes S_3 C_{ps}, \quad V_{24} = T_2 \otimes S_4 C_{ps}, \\
V_{31} &= T_3 \otimes S_1 C_{ps}, \quad V_{32} = T_3 \otimes S_2 C_{ps}, \\
V_{41} &= T_4 \otimes S_1 C_{ps}, \quad V_{42} = T_4 \otimes S_2 C_{ps}.
\end{aligned}
$$

$$4 \times 4 = 16 = 2 \times 2 \times 2 \times 2$$

Direct products:

$$V_{12} = T_1 \otimes S_2 = [N_1^T \bar{u}_{\sigma_1}(k_1) t_1 U_c \bar{u}_{\sigma_2}(k_2)] \, [N_2^S \bar{u}_{\sigma}(k_3) s_2 u_{\sigma_3}(p)]$$

• Decomposition of the wave function

$$\Phi^{\sigma}_{12\sigma_1\sigma_2\sigma_3}(1,2,3) = \Phi^{(0)\sigma}_{12\sigma_1\sigma_2\sigma_3}(1,2,3)\xi^{(0)\tau}_{\tau_1\tau_2\tau_3} + \Phi^{(1)\sigma}_{12\sigma_1\sigma_2\sigma_3}(1,2,3)\xi^{(1)\tau}_{\tau_1\tau_2\tau_3}$$

$$\Phi^{(0,1)\sigma}_{12\sigma_1\sigma_2\sigma_3}(1,2,3) = \sum_{ij} g^{(0,1)}_{ij} V_{ij}$$

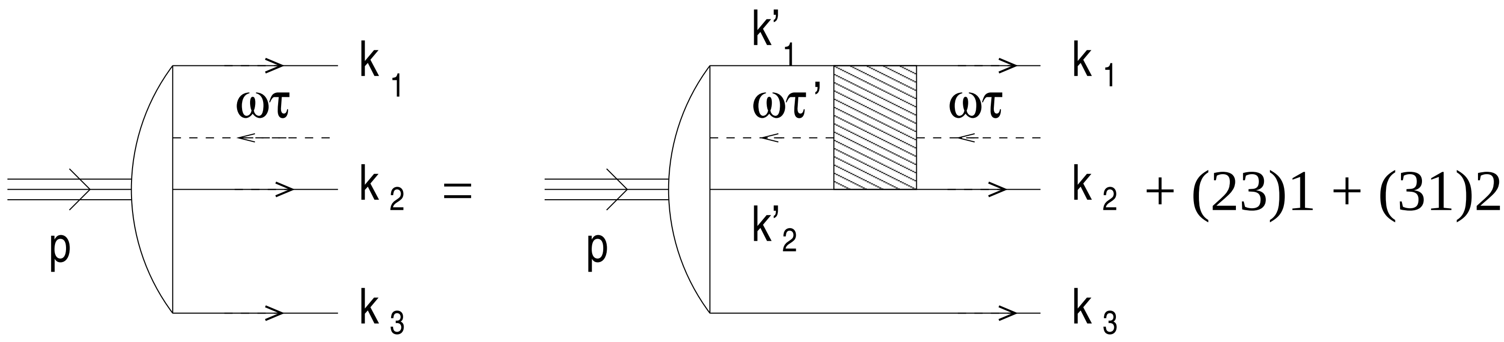
$$g^{(0,1)}_{ij} = g^{(0,1)}_{ij}(\vec{k}_{1\perp},\vec{k}_{2\perp},\vec{k}_{3\perp};x_1,x_2,x_3),$$
$$\vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{k}_{3\perp}=0;\ x_1+x_2+x_3=1.$$

16 scalar functions $g^{(0,1)}_{ij}$ depend on five scalar variables:

$$g^{(0,1)}_{ij} = g^{(0,1)}_{ij}(\vec{k}^2_{1\perp},\vec{k}^2_{2\perp},\vec{k}_{1\perp}\cdot\vec{k}_{2\perp};x_1,x_2)$$

The problem is reduced to finding them.

• System of equations



$$(\mathcal{M}^2 - M^2) g^{(n)}_{12;ij}(1,2,3) \quad \leftarrow n = 0,1 \text{ (pair isospins)}$$

$$= \sum_{i'j',n'=0,1} \int g^{(n')}_{12;i'j'}(1',2',3) W^{i'j'}_{ij}(12)(1',1;2',2;3) \frac{1}{(2\pi)^3} \frac{d^2k'_{2\perp} dx'_2}{2x'_1 x'_2}$$

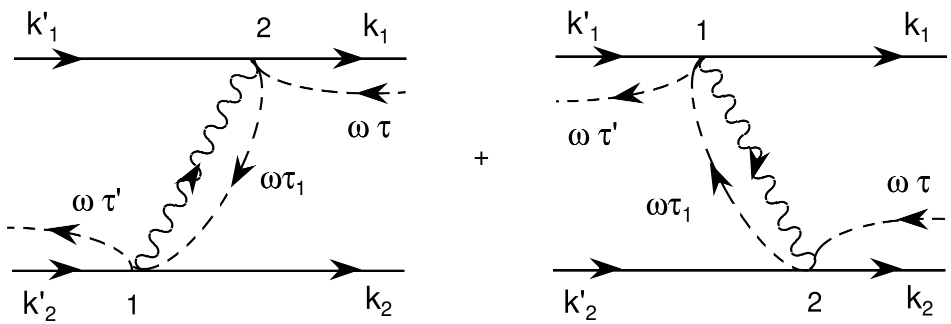
$$+ \sum_{i'j',n'=0,1} \int g^{(n')}_{12;i'j'}(3,1',2') W^{i'j'}_{ij}(31)(1',1;2',2;3) \frac{1}{(2\pi)^3} \frac{d^2k'_{2\perp} dx'_2}{2x'_1 x'_2}$$

$$+ \sum_{i'j',n'=0,1} \int g^{(n')}_{12;i'j'}(2',3,1') W^{i'j'}_{ij}(23)(1',1;2',2;3) \frac{1}{(2\pi)^3} \frac{d^2k'_{2\perp} dx'_2}{2x'_1 x'_2}$$

There is only 3D integration here: $d^2k'_{2\perp} dx'_2$!

• Interaction

The same as in the two-body system (OBE or OGE)

$\mathcal{K}$ = 

$$\mathcal{K} = \Pi_{12} O_1 O_2$$

$$W_{ij}^{i'j'}\text{(31)} = \Pi_{12} Tr\Big[(\hat{k}_2 + m) O_2 (\hat{k'}_2 + m) S_{j'}(2')(\hat{p} + M)\bar{S}_j(3)(\hat{k}_3 + m)$$
$$\times \quad T_{i'}(3, 1', 2')(-\hat{k'}_1 + m) O_1 (-\hat{k}_1 + m)\bar{T}_i(1, 2, 3)\Big]$$

• Solving system of equations

$$(\mathcal{M}^2 - M^2) g^{(n)}_{12;ij}(1,2,3) \qquad \leftarrow n = 0,1 \text{ (pair isospins)}$$

$$= \sum_{i'j',n'=0,1} \int g^{(n')}_{12;i'j'}(1',2',3) W^{i'j'}_{ij}(12)(1',1;2',2;3) \frac{1}{(2\pi)^3} \frac{d^2k'_{2\perp} dx'_2}{2x'_1 x'_2}$$

$$+ \sum_{i'j',n'=0,1} \int g^{(n')}_{12;i'j'}(3,1',2') W^{i'j'}_{ij}(31)(1',1;2',2;3) \frac{1}{(2\pi)^3} \frac{d^2k'_{2\perp} dx'_2}{2x'_1 x'_2}$$

$$+ \sum_{i'j',n'=0,1} \int g^{(n')}_{12;i'j'}(2',3,1') W^{i'j'}_{ij}(23)(1',1;2',2;3) \frac{1}{(2\pi)^3} \frac{d^2k'_{2\perp} dx'_2}{2x'_1 x'_2}$$

1. Substitute as $g^{(n)}_{12;ij}(1,2,3)$ in r.h.-side of this equation well known non-relativistic solution.
2. Iterate.

As known, the iterations converge and provide the functions $g^{(n)}_{12;ij}(1,2,3)$, determining the LFWF.

No principal difficulties: 3D numerical integral is nothing!

• Current status of the ^{3}He problem

Together with Kaiyu Fu, Zhimin Zhu and Ziqi Zhang.

The equations, especially, the kernels

$$W_{ij}^{i'j'}(31) = \Pi_{12} Tr\Big[(\hat{k}_2+m)O_2(\hat{k'}_2+m)S_{j'}(2')(\hat{p}+M)\bar{S}_j(3)(\hat{k}_3+m)$$
$$\times\ T_{i'}(3,1',2')(-\hat{k'}_1+m)O_1(-\hat{k}_1+m)\bar{T}_i(1,2,3)\Big]$$

were derived and tested. The dimension: $16\times 16\times 3 = 768$ elements, calculated numerically, by multiplication of the Dirac matrices (to avoid to derive and keep 768 matrix elements).

However, solving problem by iterations, there was found a difficulty with convergence of iterations.

The code has been debugged in oversimplified model: the three-boson spinless system with contact interaction, where the solution is known.

Hoping that the problem will be solved and the ^{3}He LFWF will be found soon!

• Tritium ^{3}H – nnp

^{3}He and ^{3}H are two isotopic states (like n and p).
However, ^{3}H is radioactive: ^{3}H $\to$ ^{3}He$+e^- + \bar{\nu}_e$.
It is rather expensive: 1 gram costs $30 000
(1000 times more expensive than gold).
1 liter of ^{3}He costs about $1000.
I don't know, whether its structure will be measured in nearest future.
However, we can make predictions.

• Nucleon

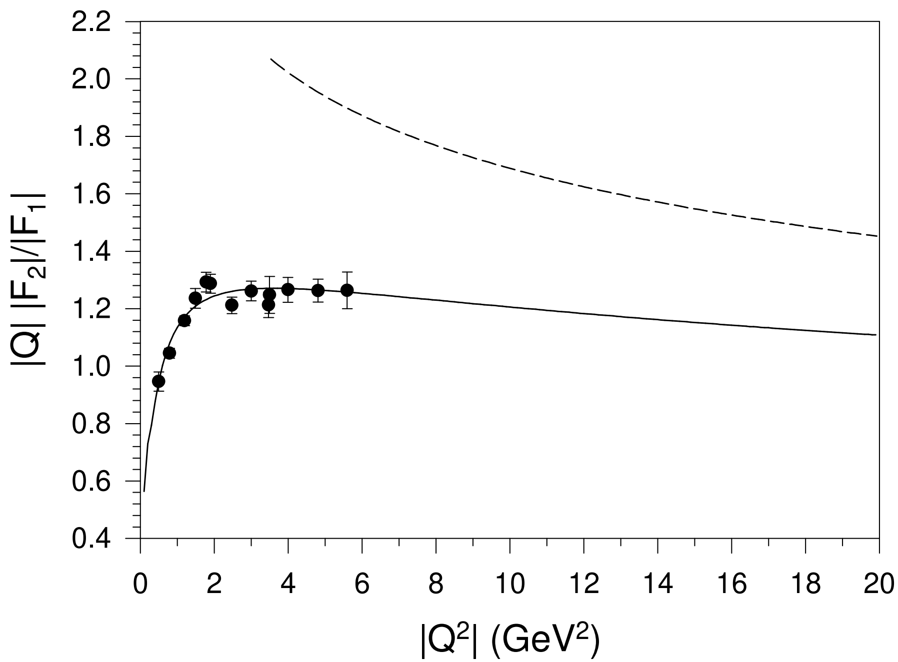
In quark model, it is also a three-fermion system.
LFWF has the same structure as for ^{3}He with the
replacement of the nucleon spinors by the quark ones and
the ^{3}He spinor by the nucleon one.
However, interaction is simpler:
exchanges by seven bosons are replaced by the
one-gluon exchange + confinement.
$SU(2)$ (isospin) symmetry of nucleons can be replaced by
$SU(3)$ symmetry of quarks.
However, the wave functions which are used so far to
describe the nucleon are oversimplified.

• Application to the nucleon ff's

S.J. Brodsky, J.R. HIller, D.S. Hwang, V.A. Karmanov,
The covariant structure of light-front wave functions and the behavior of hadronic form factors, Phys. Rev. D **69**, 076001 (2004).

$$\langle p'|J^\mu(0)|p\rangle = \bar{u}(p')\left[F_1(Q^2)\gamma^\mu + F_2(Q^2)\frac{i}{2M}\sigma^{\mu\alpha}q_\alpha\right]u(p)$$

Problem: $QF_2/F_1 \approx const$ instead of $Q^2F_2/F_1 \approx const$.



• Model

Model: di-quark ($J = 0$) + quark.

Wave function: $\psi^{\sigma}_{\sigma_1} = \bar{u}_{\sigma_1}(k_1)\left(\varphi_1 + \frac{M\hat{\omega}}{\omega\cdot p}\varphi_2\right)u^{\sigma}(p)$

Or $\psi^{\sigma}_{\sigma_1} = w^{\dagger}_{\sigma_1}\left(f_1 + \frac{i}{k}[\vec{n}\times\vec{k}]\cdot\vec{\sigma}\ f_2\right)w^{\sigma}$

Wick-Cutkosky model shows that

$$\varphi_2 = \frac{M_0 - M}{2M}\varphi_1$$

M is the nucleon mass, M_0 is the kinetic energy of the constituents.

This gives asymptotic: $QF_2/F_1 \to const$.

Yukawa model with vector exchange:

$F_2/F_1 \to \log^2(Q^2/m^2)/Q^2$ is also consistent with data.

• Towards the true nucleon LFWF

Nobody yet found and applied the nucleon LFWF with full 16 components.
Though, there is always a problem with the spin content of nucleon.
This is intriguing, important and interesting problem.
We are now ready to solve it!

• ^{4}He – nnpp

$$J^\pi = 0^+.$$

The structure of the LFWF is close the ^{3}He case: the spinor of the "initial ^{3}He is replaced by the spinor of the fourth "final" nucleon.

There are still 16 spin components. However, one extra nucleon means one extra 3D variable - complication:

$$\begin{aligned}
\psi_4 &= \psi(\vec{k}_{1\perp},\vec{k}_{2\perp},\vec{k}_{3\perp},\vec{k}_{4\perp};x_1,x_2,x_3,x_4),\\
&\quad \vec{k}_{1\perp}+\vec{k}_{2\perp}+\vec{k}_{3\perp}+\vec{k}_{4\perp}=0,\ x_1+x_2+x_3+x_4=1,\\
\to \psi_4 &= \psi(k_{1\perp},k_{2\perp},k_{3\perp},\vec{k}_{1\perp}\cdot\vec{k}_{2\perp},\vec{k}_{2\perp}\cdot\vec{k}_{3\perp};x_1,x_2,x_3)\ \leftarrow\ \text{eight scalar variables}
\end{aligned}$$

Each extra particle brings 3 extra variables, like in the non-relativistic WF.

But the integral equation is still in three variables (for two-body interaction).

• Proposals

- To calculate full nucleon LFWF
  and its em form factors
- To calculate ^{4}He LFWF and its em form factor
- Next light nuclei $A \geq 4$

^{4}He and next nuclei depend on the power of the computer facilities and availability of the working people.

The perspectives are exciting!