LF Thirring-Wess model, zero modes, abelian gauge field in 2D, two-point functions

Ľubomír Martinovič
Institute of Physics, Slovak Academy of Sciences
Dúbravská cesta 9, 845 11 Bratislava, Slovakia

August 14, 2024

ABSTRACT: An operator solution of the light-front Thirring-Wess model including the axial anomaly is briefly described. Then a novel approach to the dynamical light-front zero modes (ZMs) will be proposed. It is based on quantization of the two-dimensional LF gauge field $A^\mu(x)$ in the covariant (Feynman) gauge. The $A^\pm$ components are obtained as

a massless limit of the massive vector field and contain an infinite set of dynamical ZMs with finite LF energy. Next, a few LF "failures" are discussed. Contradictions related to the LF restriction of the two-point function are removed if the (scalar) field contains regularization terms in the plane-wave factors. As a consequence, the correct equal-LF time commutators are reproduced from the Pauli-Jordan function, and the sign function present in them is naturally replaced by a function suppressed for large values of the LF coordinate x^-. The value of the two-point function at coinciding points is aso correctly obtained in the Hamiltonian ("on-shell") formalism, contrary to claims in literature.

So far, discussion based on:

1. L. Martinovic, P. Grangé, Phys. Lett. B 724, 310 (2013)

2. L. Martinovic, P. Grangé, Few Body Syst. 56, 607 (2015)

3. L. Martinovic, *Physical and Mathematical Aspects of Symmetries*, Springer 2017, p. 253

4. L. Martinovic, A. Dorokhov, Phys. Lett. B 811, 135925 (2020)

this section:

L. Martinovic, Phys. Rev. D 107, 105009 (2023) and material presented at LC 2023, Rio de Janeiro, to be submitted

We still do not have a compact formulation of LF field theory

In particular, it has been often claimed that the light front (LF) theory has certain drawbacks, not present in the usual (conventional) "instant" form (called "space-like" (SL) here),

that it violates some essential principles - causality in DLCQ (Heinzl, Kroeger and Scheu 1999) and Lorentz invariance (N. Nakanishi and K. Yamawaki, NPB 1977, S.Tsujimaru and K. Yamawaki, PRD 1998)

even fails completely in some aspects (equal - LF time projection of two-point functions (Yamawaki)) quatization of massless fields in two space-time dimensions (G. McCartor, Z. Phys. C 1994)

new singularities for $p^+ = 0$ (B. Bakker, FBS 2011, e.g.)

very recently: Hamiltonian (on-shell, not manifestly covariant) formulation fails in the vacuum sector (P. Mannheim, P. Lowdon, S.

Brodsky, Phys. Lett. B 2020, Phys. Rev. D 2020, Phys. Repts. 2021)

A possible optimistic **explanation:** structure of the LF QFT is completely consistent and we have just not found the correct formulation of these subtle points yet (being often led by an intuition and patterns obtained in the usual SL form of QFT).

I. OPERATOR SOLUTION OF THE THIRRING-WESS MODEL IN THE LF FORMULATION

Lagrangian with the "gauge-fixing" term, close to the LF Schwinger model in the Feynman gauge

operator solution of the Heisenberg equations, vector-current conservation requires physical subspace

Correct value of the axial anomaly from the operator solution, poin-split interacting currents

The dynamics of the model is characterized by the covariant-form Lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overset{\leftrightarrow}{\partial}_{\mu} \Psi - \frac{1}{4}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} - \frac{1}{2}\big(\partial_{\mu}\tilde{B}^{\mu}\big)^{2} + \frac{1}{2}\mu^{2}\tilde{B}_{\mu}\tilde{B}^{\mu} - e\tilde{B}_{\mu}J^{\mu}, \quad (1)$$

$$\tilde{G}_{\mu\nu}(x) = \partial_{\mu}\tilde{B}_{\nu}(x) - \partial_{\nu}\tilde{B}_{\mu}(x), \quad J^{\mu}(x) = \overline{\Psi}(x)\gamma^{\mu}\Psi(x),$$

where $\Psi(x)$ and $\tilde{B}^{\mu}(x)$ are the interacting massless fermion and massive vector fields, respectively. The third term has been added (Falco 2010, e.g.) to the Lagrangian to mimic the Feynman gauge for the vector field $\tilde{B}^{\mu}$. This leads to an alternative formulation of the usual Proca version of the theory, with the same physical content (see below). Due to mathematical simplifications of the 2-dimensional dynamics, one can find an operator solution of the coupled system of the Dirac and Klein-Gordon equations:

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = e\gamma_{\mu}\tilde{B}^{\mu}(x)\Psi(x), \quad \big(\partial_{\mu}\partial^{\mu} + \mu^{2}\big)\tilde{B}^{\nu}(x) = eJ^{\nu}(x). \qquad (2)$$

The latter emerged from the Proca equation due to the presence of the "gauge-fixing" term in the Lagrangian (1). The LF form of that Lagrangian:

$$\mathcal{L} = i\Psi_2^\dagger \overset{\leftrightarrow}{\partial}_+ \Psi_2 + i\Psi_1^\dagger \overset{\leftrightarrow}{\partial}_- \Psi_1 - 2\partial_+\tilde{B}^+\partial_-\tilde{B}^- + \frac{1}{2}\mu^2\tilde{B}^+\tilde{B}^- - \frac{e}{2}\tilde{B}^+J^- - \frac{e}{2}\tilde{B}^-J^+, \tag{3}$$

with the corresponding coupled LF field equations

$$2i\partial_+\Psi_2(x) = e\tilde{B}^-(x)\Psi_2(x), \tag{4}$$

$$2i\partial_-\Psi_1(x) = e\tilde{B}^+(x)\Psi_1(x), \tag{5}$$

$$\left(4\partial_+\partial_- + \mu^2\right)\tilde{B}^+(x) = eJ^+(x), \tag{6}$$

$$\left(4\partial_+\partial_- + \mu^2\right)\tilde{B}^-(x) = eJ^-(x). \tag{7}$$

The classical solution of the Dirac equation involves the free massless LF fermion field components $\psi_1(x^+)$ and $\psi_2(x^-)$:

$$\Psi_1(x) = e^{-\frac{ie}{2}\int\limits_{-\infty}^{+\infty} dy^- \frac{1}{2}\epsilon(x^- - y^-)\tilde{B}^+(x^+, y^-)} \psi_1(x^+), \tag{8}$$

$$\Psi_2(x) = e^{-\frac{ie}{2}\int\limits_{-\infty}^{+\infty} dy^+ \frac{1}{2}\epsilon(x^+ - y^+)\tilde{B}^-(y^+, x^-)} \psi_2(x^-). \tag{9}$$

$\epsilon(x^\pm)$ is the sign function, $\partial_\pm \epsilon(x^\pm) = 2\delta(x^\pm)$. The free fermion fields have the Fock representation

$$\psi_2(x^-) = \int\limits_0^{+\infty} \frac{dp^+}{\sqrt{4\pi}} \left[b(p^+) e^{-\frac{i}{2}p^+ x^-} + d^\dagger(p^+) e^{\frac{i}{2}p^+ x^-} \right], \tag{10}$$

$$\{b(p^+), b^\dagger(q^+)\} = \{d(p^+), d^\dagger(q^+)\} = \delta(p^+ - q^+), \tag{11}$$

$$\psi_1(x^+) = \int_0^{+\infty} \frac{dp^-}{\sqrt{4\pi}} \left[\tilde{b}(p^-)e^{-\frac{i}{2}p^- x^+} - \tilde{d}^\dagger(p^-)e^{\frac{i}{2}p^- x^+}\right], \tag{12}$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-). \tag{13}$$

These fields satisfy the massless LF Dirac eq. $\partial_+\psi_2 = 0,\ \partial_-\psi_1 = 0$ in two dimensions and have been obtained as massless limits of the corresponding massive fields. The two-point functions calculated from $\psi_1(x^+)$ and $\psi_2(x^-)$ coincide with the massless limits of the two-point functions of the massive fields $\psi_1(x^+, x^-)$ and $\psi_2(x^+, x^-)$. This confirms the consistency of the above quantization rules. It follows from the Fock anticommutation relations

(11),(13) that the massless fields satisfy

$$\{\psi_1(x^+), \psi_1^\dagger(y^+)\} = \delta(x^+ - y^+), \tag{14}$$

$$\{\psi_2(x^-), \psi_2^\dagger(y^-)\} = \delta(x^- - y^-). \tag{15}$$

Note that the LF massless fermion fields (10),(12) depend on only one variable $x^\pm$ while in the previous treatments (McCartor 1994) one assumed $\psi_1(x^+, x^-)$ and $\psi_2(x^+, x^-)$.

The interacting current appearing in the Klein-Gordon equations can be determined from the solution (8),(9) by means of the point-split regularized definition

$$J^+(x) = \lim_{\epsilon \to 0} \Big[\Psi_2^\dagger(x + \frac{\epsilon}{2})\Psi_2(x - \frac{\epsilon}{2}) + H.c.\Big], \tag{16}$$

$$J^-(x) = \lim_{\epsilon \to 0} \Big[\Psi_1^\dagger(x + \frac{\epsilon}{2})\Psi_1(x - \frac{\epsilon}{2}) + H.c.\Big]. \tag{17}$$

Using the free-field relations

$$\psi_2{}^\dagger(x^- + \frac{\epsilon^-}{2})\psi_2(x^- - \frac{\epsilon^-}{2}) =: \psi_2{}^\dagger(x^-)\psi_2(x^-) : -V(\epsilon^-),$$
$$\psi_1{}^\dagger(x^+ + \frac{\epsilon^+}{2})\psi_1(x^+ - \frac{\epsilon^+}{2}) =: \psi_1{}^\dagger(x^+)\psi_1(x^+) : -V(\epsilon^+),$$
$$V(\epsilon^\pm) = \frac{i}{2\pi}\frac{1}{\epsilon^\pm - i\eta}, \tag{18}$$

one arrives at

$$J^+(x) = j^+(x^-) + \frac{e}{2\pi}\int\limits_{-\infty}^{+\infty} dy^+ \epsilon(x^+ - y^+)\partial_- \tilde{B}^-(y^+, x^-), \tag{19}$$

$$J^-(x) = j^-(x^+) + \frac{e}{2\pi} \int_{-\infty}^{+\infty} dy^- \epsilon(x^- - y^-) \partial_+ \tilde{B}^+(x^+, y^-), \qquad (20)$$

where the free current is given by

$$j^+(x^-) = 2 : {\psi_2}^\dagger(x^-) \psi_2(x^-) :, \quad j^-(x^+) = 2 : \psi_1^\dagger(x^+) \psi_1(x^+) : . \qquad (21)$$

In a physically sensible theory, the vector current has to be conserved, $\partial_\mu J^\mu(x) = 0$. From the expressions (19), (20) we find

$$\partial_\mu J^\mu(x) = \partial_+ J^+(x) + \partial_- J^-(x) = -\frac{e}{\pi} \big(\partial_+ \tilde{B}^+ + \partial_- \tilde{B}^- \big). \qquad (22)$$

In the usual Proca form of the theory (without the "gauge-fixing" term in th Lagrangian), the equation $\partial_\mu \tilde{B}^\mu = 0$ is a strong (operator) relation following from the field equations. Here we have to impose it as a condition defining the physical subspace,

$$\partial_\mu \tilde{B}^{\mu(+)}(x)|phys\rangle = 0, \tag{23}$$

where $\tilde{B}^{\mu(+)}(x)$ is the positive-frequency (annihilation) part of the field. This ensures conservation of the vector current in the physical subspace of the model. On the other hand, the divergence of the axial-vector current

$$J_5^\mu(x) = \overline{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) = \big(J^+(x), -J^-(x)\big) \tag{24}$$

is non-zero ("anomalous"),

$$\partial_\mu J_5^\mu = -\frac{e}{\pi}\big(\partial_+\tilde{B}^+ - \partial_-\tilde{B}^-\big) = \frac{e}{2\pi}\epsilon_{\mu\nu}\tilde{G}^{\mu\nu}. \tag{25}$$

$\epsilon^{\mu\nu}$ is the antisymmetric tensor with $\epsilon^{+-} = -2$. One can see that the axial anomaly is a purely quantum effect and is obtained here non-perturbatively, directly from the exact operator solution of the field equations. This result is in agreement with the recent LF study of the model (L. Martinovic, PRD '23) in the Proca version, in which $\partial_+ B^+(x) + \partial_- B^-(x) = 0$ holds as an operator relation. The rest of the analysis can proceed analogously to (LM, PRD'23) . On the physical subspace, the interacting current takes after partial integration the form

$$J^+(x) = j^+(x^-) - \frac{e^2}{\pi} B^+(x), \quad J^-(x) = j^-(x^+) - \frac{e^2}{\pi} B^-(x), \tag{26}$$

so that the field equations become

$$\left(4\partial_+\partial_- + \tilde{\mu}^2\right)\tilde{B}^+(x) = e j^+(x^-), \tag{27}$$

$$\left(4\partial_+\partial_- + \tilde{\mu}^2\right)\tilde{B}^-(x) = e j^-(x^+), \quad \tilde{\mu}^2 = \mu^2 + \frac{e^2}{\pi}. \tag{28}$$

Their solution is

$$\tilde{B}^+(x) = B^+(x;\tilde{\mu}) + \frac{e}{\tilde{\mu}^2} j^+(x^-), \quad \tilde{B}^-(x) = B^-(x;\tilde{\mu}) + \frac{e}{\tilde{\mu}^2} j^-(x^+). \tag{29}$$

The fields $B^\mu(x;\tilde{\mu})$ satisfy the Klein-Gordon equation with the renormalized mass $\tilde{\mu}$. It follows from (26) that the condition (23) reads actually $\partial_\mu B^{\mu(+)}(x)|phys\rangle = 0$. Two-point functins computed, etc.

II. MASSLESS LF SCALAR FIELD, $A^\mu(x)$ in 2D, AND A FRESH LOOK AT LF ZERO MODES IN 2D

constrained and **dynamical** zero modes (review: M. Burkardt, Adv. Nucl. Phys. 1996)

periodic boundary conditions for finite $L, L_\perp$

$\approx$ Fourier modes with $p^+ = 0$: $\phi(x) = \phi_0(x^+, x_\perp) + \phi_N(x^+, x^-, x_\perp)$

LF scalar field (Nakanishi and Yamawaki, NPB 1977, McCartor and Robertson, Z. Phys. C 1992):

$$\partial_\mu \partial^\mu = 4\partial_+ \partial_- - \partial_\perp^2 \Rightarrow \phi_0(x^+, x_\perp) = \frac{\lambda}{\mu^2} \int\limits_{-L}^{+L} \frac{dx^-}{2L} \big(\phi_0 + \phi_n\big)^3$$

ϕ_0 is a non-linear operator function of all normal modes

LF SSB in $\lambda\phi^4(1+1)$ (Pinsky, van de Sande and Hiller 1995)

DYNAMICAL ZM: independent Fourier modes, QED(3+1) in the (modified) LC gauge $A_N^+(x) = 0$, proper ZMs $a^\mu(x^+, x_\perp)$ constrained, $A_0^+(x^+)$ satisfies dynamical equation

$$\partial_+^2 A_0^+ = eJ_0^-$$

usually: A_0^+ intepreted as QM variable, "vacuum potential" (non-abelian models, A. Kalloniatis, PRD 1996, e.g.)

Relation to the second-quantized picture where (naively)

$p^- = \frac{p_\perp^2 + m^2}{p^+} = \infty$?

$A^+(x^+)$ is just one mode, but what is its LF energy ? $(0/0)$

Quantization of massless LF fields in $D = 1+1$ - a conceptual problem for a few decades. Some degrees of freedom seemed to be missing and had to be introduced by hand (Heinzl, Krusche and Werner PLB 1992, McCartor ZPC 1994, McCartor, Pinsky and Robertson PRD 1996)

a consistent quantization scheme for massless scalar and fermion fields, based on the corresponding massive theories (Martinovic and Grangé).

scalar field, quantum solution of the massive LF Klein-Gordon equation

$$\left(4\partial_+\partial_- + \mu^2\right)\phi(x) = 0$$

given by the expansion (71),

$$\phi(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \Big[a(k^+) e^{-\frac{i}{2}k^+(x^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(x^+ - i\epsilon^+)} + \\ + a^\dagger(k^+) e^{\frac{i}{2}k^+(x^- + i\epsilon^-) + \frac{i}{2}\frac{\mu^2}{k^+}(x^+ + i\epsilon^+)} \Big], \tag{30}$$

$$\big[a(k^+), a^\dagger(l^+)\big] = \delta(k^+ - l^+), \ \big[a(k^+), a(l^+)\big] = 0. \tag{31}$$

If $\mu = 0$, the form of the field equation and its solution is

$$\partial_+\partial_-\phi_0(x) = 0, \quad \phi_0(x) = \varphi(x^-) + \varphi_0(x^+). \tag{32}$$

In the classical case, the functions $\varphi_0(x^+)$ and $\varphi(x^-)$ have usually been considered to be arbitrary (Yan, McCartor and Robertson)

However, setting $\mu = 0$ in the quantum solution (30) directly yields $\varphi(x^-)$ with the same Fock algebra (73) (infrared cutoff η necessary):

$$\varphi(x^-) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+) e^{-\frac{i}{2}k^+x^-} + a^\dagger(k^+) e^{\frac{i}{2}k^+x^-} \right]. \tag{33}$$

For symmetry reasons, it is natural to expect a similar solution for $\varphi_0(x^+)$. The change of variables $k^+ \to k^- = \frac{\mu^2}{k^+}$ in the field expansion (71) indeed gives for $\mu = 0$ (LM, PRD 2023)

$$\varphi_0(x^+) = \int_0^\infty \frac{dk^-}{\sqrt{4\pi k^-}} \left[\tilde{a}(k^-) e^{-\frac{i}{2}k^-x^+} + \tilde{a}^\dagger(k^-) e^{\frac{i}{2}k^-x^+} \right],$$

$$\left[\tilde{a}(k^-), \tilde{a}^\dagger(l^-) \right] = \delta(k^- - l^-), \ \left[\tilde{a}(k^-), a^\dagger(l^+) \right] = 0. \tag{34}$$

Existence of this component is mandatory:

IMPORTANT: The two-point functions calculated from the fields (33) and (34) coincide with the massless limits of the two-point functions calculated from the massive scalar field - consistency check

In particular, the two-point function ($z = x - y$)n

$$D_2^{(+)}(x-y) = 2\langle 0|\phi(x)\partial_+\phi(y)|0\rangle = \theta(-z^2)\frac{\mu}{2\pi}\sqrt{-\frac{z^-}{z^+}}K_1(\mu\sqrt{-z^2}) +$$
$$+\theta(z^2)\frac{\mu}{4}\sqrt{\frac{z^-}{z^+}}\Big[Y_1(\mu\sqrt{z^2}) + i\epsilon(z^+)J_1(\mu\sqrt{z^2})\Big] \tag{35}$$

has a non-vanishing massless limit

$$D_2^{(+)}(x-y;\mu=0) = \frac{1}{4\pi}\frac{1}{x^+ - y^+ - i\epsilon^+}. \tag{36}$$

Implication: there must exist a massless ZM field yielding (36) directly. Eq.(34) is that field since $\langle 0|\varphi_0(x^+)\partial_+\varphi_0(y^+)|0\rangle$ exactly reproduces (36).

$k^+ = k^-$, in analogy to the SL dispersion relation $k^0 = |k^1|$.

Same for the massless limit of $\langle 0|\phi(x)\partial_-\phi(y)|0\rangle$ and $\langle 0|\varphi(x^-)\partial_-\varphi(y^-)|0\rangle$ computed from $\varphi(x^-)$ (33)

A pattern similar to the scalar-field case can as well be expected for the LF gauge field $A^\mu(x)$ because of its masslessness

The gauge invariance of the free Lagrangian $\mathcal{L} = -1/4F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, under the transformations $A^\mu(x) \to A^\mu(x) - \partial^\mu\Lambda(x)$

$$A^+(x^\pm) \to A^+(x^\pm) - 2\partial_-\Lambda(x^\pm), \quad A^-(x^\pm) \to A^-(x^\pm) - 2\partial_+\Lambda(x^\pm) \quad (37)$$

suggests that 2 components out of four $A^\pm(x^\pm)$ can be eliminated leaving

$A^+(x^+)$ and $A^-(x^-)$ as physical fields. The detailed analysis best performed in the **covariant** (Feynman) gauge with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\big(\partial_\mu A^\mu\big)^2 = -2\partial_+ A^+ \partial_- A^-. \tag{38}$$

The solution of the associated 2D Maxwell equation $\quad \partial_+\partial_- A^\pm(x) = 0$ should also consist of pieces that depend on x^+ or x^- SEPARATELY.

To quantize the two-dimensional LF gauge field consistently – useful to view its 2 components as **a massless limit of the corresponding massive field** and therefore to add a mass perturbation to the Lagrangian (38):

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\lambda^2 B_\mu B^\mu - \frac{1}{2}(\partial_\mu B^\mu)^2, \tag{39}$$

where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. In the LF form, we have

$$\mathcal{L} = -2\partial_+ B^+ \partial_- B^- + \frac{\lambda^2}{2} B^+ B^- \Rightarrow \left(4\partial_+\partial_- + \lambda^2\right) B^\pm(x) = 0. \qquad (40)$$

$B^\pm(x)$ satisfy the 2-dimensional Klein-Gordon equation. The components of the energy-momentum tensor are

$$T^{++} = -4\partial_- B^- \partial_- B^+, \quad T^{+-} = -4\lambda^2 B^+ B^-. \qquad (41)$$

The conjugate momenta $\Pi^\mu = \delta\mathcal{L}/\delta\partial_+ B_\mu$ are $\Pi^+ = 0$, $\Pi^- = -4\partial_- B^-$. Unlike the SL theory, the "gauge-fixing term" in the Lagrangians (38,40) did not generate the non-vanishing momentum of the field $B^-(x)$.

The covariant form of the ETCR leads to

$$\left[B^+(x^+, x^-), \Pi^-(x^+, y^-)\right] = ig^{+-}\delta(x^- - y^-), \qquad (42)$$

$g^{+-} = 2$. The equations (40) suggest the Fock expansion

$$B^+(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \big[c(k^+) e^{-i\hat{k}\cdot x} + c^\dagger(k^+) e^{i\hat{k}\cdot x} \big], \tag{43}$$

$$\big[c(k^+), c^\dagger(l^+) \big] = \delta(k^+ - l^+), \tag{44}$$

$$\hat{k}\cdot x \equiv \frac{1}{2} k^+ x^- + \frac{1}{2} \frac{\lambda^2}{k^+} x^+,$$

analogous to the scalar field (71). Remarkably, the correct Fock form of the energy and momentum operators

$$P^\mu = \int_0^{+\infty} dk^+ \hat{k}^\mu c^\dagger(k^+) c(k^+), \quad \hat{k}^\mu = (k^+, \frac{\lambda^2}{k^+}). \tag{45}$$

obtained from the densities (41) only if $B^-(x) = -B^+(x)$. This condition reduces the number of independent field variables to one in accord with the conventional Proca theory, where the operator relation $\partial_\mu B^\mu = 0$ follows from the antisymmetry of $G^{\mu\nu}$ and takes care of the reduction.

The Lagrangian (40) and ETCR acquire the scalar-field form

$$\mathcal{L} = 2\partial_+ B^+ \partial_- B^+ - \frac{1}{2}\lambda^2 B^+ B^+, \quad \Pi^- = 4\partial_- B^+. \tag{46}$$

The advantage of the present formulation of the LF massive vector field: its massless limit is non-singular. As in the scalar-field case:

$$B^+(x, \lambda = 0) \equiv A^+(x) = A^+(x^-) + A_0^+(x^+),$$

where

$$A^+(x^-) = \int\limits_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \left[c(k^+)e^{-\frac{i}{2}k^+x^-} + c^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right],$$

$$\left[c(k^+), c^\dagger(l^+)\right] = \delta(k^+ - l^+), \quad (47)$$

$$A_0^+(x^+) = \int\limits_0^\infty \frac{dk^-}{\sqrt{4\pi k^-}} \left[\tilde{c}(k^-)e^{-\frac{i}{2}k^-x^+} + \tilde{c}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right],$$

$$\left[\tilde{c}(k^-), \tilde{c}^\dagger(l^-)\right] = \delta(k^- - l^-), \quad \left[c(k^+), \tilde{c}^\dagger(k^-)\right] = 0. \quad (48)$$

The resulting Lagrangian $\mathcal{L} = 2\partial_+ A^+ \partial_- A^+$ has no residual gauge freedom

P^μ operators are

$$P^+ = \int_{-\infty}^{+\infty} dx^- 2\big(\partial_- A^+(x^-)\big)^2 = \int_0^{+\infty} dk^+ k^+ c^\dagger(k^+) c(k^+), \tag{49}$$

$$P^- = \lim_{\lambda\to 0} \lambda^2 \int_{-\infty}^{+\infty} dx^- \big(B^+(x)\big)^2 = \int_0^{+\infty} dk^+ k^- \tilde{c}^\dagger(k^+) \tilde{c}(k^+) \tag{50}$$

Any state of the form $\tilde{c}^\dagger(k_1^-)|0\rangle$, $\tilde{c}^\dagger(k_1^-)\tilde{c}^\dagger(k_2^-)|0\rangle$, ... containing the ZM quanta, has finite LF energy but vanishing LF momentum. For example, based on above Fock CR,

$$P^- \tilde{c}^\dagger(k_1^-)|0\rangle = k_1^- \tilde{c}^\dagger(k_1^-)|0\rangle, \quad P^+ \tilde{c}^\dagger(k_1^-)|0\rangle = 0, \tag{51}$$

so that $M^2\tilde{c}^\dagger(k_1^-)|0\rangle = 0, M^2 = P^+P^-$. Vacuum degeneracy?

Remark: field equations prohibit modes with $k^+ = 0$ for massive fields $\mu^2\phi_0(x^+) = 0$

Also: $k^+ = 0$ modes of LF Feynman diagrams are not the genuine LF zero modes

integration $-\infty < k^\pm < +\infty \Rightarrow \delta(k^+)$ contribution after dk^-, not available in LF perturbation theory, $p^\pm > 0$

LF QED(3+1) in the Feynman gauge The above gauge fixing and zero-mode analysis can be generalized to the realistic QED(3+1) theory. In the LF literature, the light-cone or light-front gauge $A^+(x) = 0$ has been the typical choice of gauge (Kogut and Soper 1970, Kalloniatis and Pauli 1994). A more detailed analyses, performed in the finite volume with (anti)periodic boundary conditions, revealed the physical gauge degree of freedom - the zero mode $A_0^+(x^+)$, in addition to the constrained "proper zero modes"

For simplicity, we shall consider the free electrodynamics with the 4-dimensional version of the Lagrangian (38) (Mannheim, PRD 2020):

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\big(\partial_\mu A^\mu\big)^2, \\ \mathcal{L}_{lf} &= \frac{1}{2}\big[(\partial_+ A^+ - \partial_- A^-)^2 - (\partial_1 A^2 - \partial_2 A^1)^2 + \\ &\quad + (2\partial_+ A^i + \partial_i A^-)(2\partial_- A^i + \partial_i A^+) - \\ &\quad - (\partial_+ A^+ + \partial_- A^- + \partial_i A^i)^2\big]. \end{aligned} \tag{52}$$

Here the index $i = 1, 2$ and we will also use the notation $\partial_\perp^2 = \partial_1^2 + \partial_2^2$, so that $\partial_\mu \partial^\mu = 4\partial_+\partial_- - \partial_\perp^2$. The gauge invariance of the above Lagrangian is restricted to

$$A^\mu(x) \to A^\mu(x) - \partial^\mu \Lambda(x), \quad \partial_\mu \partial^\mu \Lambda(x) = 0, \tag{53}$$

i.e. the gauge function is not arbitrary but must obey the above equation. The Euler-Lagrange equations without the gauge-fixing term in $\mathcal{L}$, read

$$(2\partial_+\partial_- - \partial_\perp^2)A^+ - 2\partial_-\big(\partial_- A^- + \partial_i A^i\big) = 0, \tag{54}$$

$$(2\partial_+\partial_- - \partial_\perp^2)A^- - 2\partial_+\big(\partial_+ A^+ + \partial_i A^i\big) = 0, \tag{55}$$

$$(4\partial_+\partial_- - \partial_\perp^2)A^i + \partial_i(\partial_+ A^+ + \partial_- A^- + \partial_j A^j) = 0. \tag{56}$$

The gauge-fixing piece adds a term $\partial^\mu(\partial_+ A^+ + \partial_- A^- + \partial_i A^i)$ to each corresponding equation, leading to

$$(4\partial_+\partial_- - \partial_\perp^2)A^\mu(x) = 0. \tag{57}$$

This result relies on an implicit assumption that the gauge field depends on all three space variables x^-, x^1, x^2 (the "normal-mode sector"). If one assumes existence of the x^--independent field components $a^\mu(x^+, x_\perp)$, only the terms without the derivative ∂_- survive in the Lagrangian (52), leading to the equtions

$$\partial_\perp^2 a^\mu(x^+, x_\perp) = 0, \tag{58}$$

corresponding to the "proper zero-mode sector". In an interacting theory (say, if there is a non-dynamical source $J^\mu(x)$ on the rhs of the equations (56), the latter equations express the proper zero modes in term of J^μ, because the inverse derivative $\partial_\perp^{-2}$ is well defined. On the other hand, the Lagrangian (52) does not contain the $global$ zero-mode components (fields independent on both $x^-, x_\perp$) except for the $\partial_+ A^+$ and $\partial_+ A^i$ terms which coincide with the 2-dimensional theory, leading to the field equation

$$\partial_+ \partial_- A^\mu(x^+, x^-) = 0 \tag{59}$$

and to quantization which has to start from the massive field as described in the previous paragraphs.

One can proceed in the canonical quantization and also derive the LF Hamiltonian. Here we shall merely note that the covariant equal-LF time commutation relations

$$\left[A^\mu(x^+,\underline{y}),\Pi^\nu(x^+,\underline{y})\right] = ig^{\mu\nu}\delta^{(3)}(\underline{x}-\underline{y}), \tag{60}$$

where $\underline{x} \equiv (x^-, x^1, x^2)$, imply

$$\left[A^+(x^+,\underline{x}),\Pi^-(x^+,\underline{y})\right] = ig^{+-}\delta^{(3)}(\underline{x}-\underline{y}), \tag{61}$$

$$\left[A^i(x^+,\underline{x}),\Pi^j(x^+,\underline{y})\right] = ig^{ij}\delta^{(3)}(\underline{x}-\underline{y}). \tag{62}$$

Then, with $g^{+-} = 2, g^{11} = g^{22} = -1, g^{12} = 0$, and with

$$\Pi^\mu = \frac{\delta \mathcal{L}}{\delta \partial_+ A_\mu}, \quad \Pi^+(x) = 0, \tag{63}$$

$$\Pi^-(x) = -4\partial_- A^-(x) + 2\partial_i A^i(x), \tag{64}$$

$$\Pi^i(x) = -2\partial_- A^i(x) - \partial_i A^+(x) \tag{65}$$

one arrives at the equal-time commutators of the scalar-field type

$$\left[A^+(x^+, \underline{x}), 2\partial_- A^+(x^+, \underline{y})\right] = i\delta^{(3)}(\underline{x} - \underline{y}), \tag{66}$$

$$\left[A^1(x^+, \underline{x}), 2\partial_- A^1(x^+, \underline{y})\right] = i\delta^{(3)}(\underline{x} - \underline{y}), \tag{67}$$

$$\left[A^2(x^+, \underline{x}), 2\partial_- A^2(x^+, \underline{y})\right] = i\delta^{(3)}(\underline{x} - \underline{y}). \tag{68}$$

In obtaining these commution relations, the usual assumption $\left[A^\mu, A^\nu\right] = 0$, if $\mu \neq \nu$, was made. In addition, like in the 2-dimensional theory, the relation $A^{-}(x) = -A^{+}(x)$, is required for consistency and the correct form of Poincaré generators. In the conventional SL quantization in the covariant gauge, the A^0 field component acquires a conjugate momentum $-\partial_\mu A^\mu$, but considering the latter as an operator equal to zero contradicts the canonical commutation relations. Instead, one has to require $\partial_\mu A^{\mu(+)}(x)|phys\rangle = 0$ as a condition on physical states, along with introduction of the indefinite-metric Hilbert space (the Gupta-Bleueler quantization). In the present LF formulation, no such construction is necessary: the gauge-fixing term does not supply the gauge-field component A^- with the conjugate momentum ($\Pi^+ = 0$ in (63)), that is it remains to be a non-dynamical quantity, but the required relation $A^-(x) = -A^+(x)$ resolves this apparent paradox without the need to introduce the indefinite metric. Moreover, the residual gauge freedom of the Lagrangian (52) is fully removed ("fixed") by this condition.

III. LIGHT-FRONT RESTRICTION OF THE TWO-POINT AND COMMUTATOR FUNCTIONS

A claim of non-existence of light-front quantized field theory made long time ago by Nakanishi and Yamawaki

The alleged trouble was an incorrect - mass-independent - form of the two-point function of the massive scalar field restricted to the LF $x^+ = 0$

reason: setting $x^+ = 0$ in the scalar field expansion kills the mass dependence due to

$\exp\{-\frac{i}{2}\frac{k_\perp^2+\mu^2}{k^+}x^+\}$, no mass in $\frac{dk^+}{k^+}$ in contrast to

$\exp\{-iE(k)t\}$ and mass in $\frac{d^3k}{2E(k)}$

straightforward application of equal-time commutation relations (ETCR) also seemed to generate inconsistencies in the interacting theory within the Kallén-Lehmann representation. Contradictions disappear if a careful mathematical treatment is applied. The central quantities - the 2-point function $D^{(+)}(x-y)$ of the massive scalar field $\phi(x)$ and the related Pauli-Jordan function $D(x-y)$:

$$iD^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle, \tag{69}$$

$$iD(x-y) = iD^{(+)}(x-y) - iD^{(+)}(y-x). \tag{70}$$

2D case first for simplicity. Our field expansion

$$\phi(x) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}}\big[a(k^+)e^{-\frac{i}{2}k^+(x^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(x^+ - i\epsilon^+)} + \tag{71}$$

$$+ a^\dagger(k^+)e^{\frac{i}{2}k^+(x^- + i\epsilon^-) + \frac{i}{2}\frac{\mu^2}{k^+}(x^+ + i\epsilon^+)}\big], \tag{72}$$

$$\left[a(k^+), a^\dagger(l^+)\right] = \delta(k^+ - l^+), \ \left[a(k^+), a(l^+)\right] = 0 \tag{73}$$

differs from the conventional one by the convergence factors $\exp(-k^+\epsilon^-)$ and $\exp(-\mu^2\epsilon^+/k^+)$. A straightforward calculation gives

$$iD^{(+)}(z) = \int_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\epsilon^+)}. \tag{74}$$

Here $z = x - y$. Small imaginary parts of the arguments $z^\pm$ necessary for the existence of the above integral (see Gradshteyn and Ryzhik, e.g.)

explicit evaluation yields for $x^+ > 0$

$$iD^{(+)}(x) = \frac{\theta(-x^2)}{2\pi} K_0\big(\mu\sqrt{-x^2}\big) + \frac{\theta(x^2)}{4i} H_0^{(2)}\big(\mu\sqrt{x^2}\big),$$
$$H_0^{(2)}(z) = J_0(z) - iY_0(z). \tag{75}$$

$H_\nu^{(2)}(x), J_\nu(x)$, $Y_\nu(x)$ and $K_\nu(x)$ are Bessel functions with $\pm i\epsilon^\pm$ being implicitly present. The LF restriction of the correlation function for $x^2 < 0$

$$D^{(+)}(x^+ = 0, x^-) = \frac{1}{2\pi} K_0\big(\mu\sqrt{-i\epsilon^+ x^-}\big). \tag{76}$$

Coincides with the corr function calculated from two scalar fields restricted to the LF, whose Fock expansion is given by setting $x^+ = 0$ in (71).

In the previous treatments, different results obtained depending on whether one sets $x^+ = 0$ in the calculated two-point function or computes this function from the fields taken at non-zero $\epsilon^\pm$.

In the time-like region, the commutator function for unequal times is

$$iD(z) = \frac{1}{4i} H_0^{(2)}\Big(\mu\sqrt{(z^+ - i\epsilon^+)(z^- - i\epsilon^-)}\Big) - c.c.. \tag{77}$$

For finite $x^- - y^-$, $iD(x-y)$ reduces to ETCR

$$\big[\phi(x^+,x^-),\phi(x^+,y^-)\big] = -\frac{i}{4}\epsilon(x^- - y^-), \tag{78}$$

where $\epsilon(x) = x/|x|$ is the sign function. This follows from

$$iD^{(+)}(0,z^- > 0) = \langle 0|\phi(x^+,x^-)\phi(x^+,y^-)|0\rangle =$$
$$= \int\limits_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{1}{2}\frac{\mu^2}{k^+}\epsilon^+} = \frac{1}{4i} H_0^{(2)}(\mu\sqrt{-i\epsilon^+ z^-})$$
$$= -\frac{\gamma_E}{2\pi} - \frac{1}{4\pi}\ln\Big(\frac{\mu^2 z^-}{4}\epsilon^+\Big) - \frac{i}{8}, \tag{79}$$

inserted into (77) taken at $x^+ = y^+$. The result is $-i/4$. For $z^- < 0$, the complex conjugate results in (79) and (77) found. In obtaining the (79), the expansions $J_0(x) \approx 1 + O(x^2)$, $Y_0(x) \approx \frac{2}{\pi}\big[\gamma_E + \ln\frac{x}{2}\big]$ valid for $x \ll 1$ used along with the relation $\ln(i) = i\pi/2$. γ_E is the Euler's constant.

Introduction of $\epsilon^+ \neq 0$ regulates the logarithmic divergence in (79) and simultaneously ensures the correct value of the ETCR (78).

A similar derivation can be given for the fermion field

The correctness of the commutator function (77) manifests itself also for large values of its argument because in that domain $D^{(+)}(z)$ is actually damped as follows from the asymptotic expansion for $x \to \infty$

$$H_0^{(2)}(x) \approx \frac{2}{\sqrt{\pi x}} \exp\big(-i(x - \frac{\pi}{4})\big), \tag{80}$$

leading to the behaviour $\sim (\epsilon^+ z^-)^{-1/4} \exp\big(-\frac{\mu}{2}\sqrt{\epsilon^+ z^-}\big)$ for each of the two terms in the limit $z^- \to \infty$. Consequently, the commutator function at

$z^+ = 0$ does not reduce to the sign function for large z^- separations but is exponentially suppressed.

helpful for suppresion of surface terms in the LF Poincaré algebra and covariance relations

In the **interacting** theory, the Kallén-Lehmann representation for the correlation function $\hat{D}^{(+)}(x)$ of the interacting field is valid in the axiomatic framework (Streater and Wghtman, Tsujimaru and Yamawaki)

$$\hat{D}^{(+)}(x) = \int_0^\infty d\kappa^2 \rho(\kappa^2) D^{(+)}(x;\kappa^2), \tag{81}$$

where $\rho(\kappa^2)$ is the spectral function. In the formulation without $\epsilon^\pm$ regularization - a **contradiction**: both $\hat{D}^{(+)}(x)$ and $\hat{D}(x)$ coincide with their FREE counterparts at $x^+ = 0$. In our regularized approach, this difficulty removed because the $\hat{D}^{(+)}(x)$ function does depend on κ even at $x^+ = 0$:

$$\hat{D}^{(+)}(0, x^-) = \frac{1}{2\pi} \int_0^\infty d\kappa^2 \rho(\kappa^2) K_0(\kappa\sqrt{-i\epsilon^+ x^-}). \tag{82}$$

The same conclusion is valid also for the interacting commutator function $\hat{D}(x)$. It follows that the free and interacting theories differ fundamentally also in the LF form of the relativistic dynamics and the no-go theorem found by Yamawaki and collaborators is not valid.

Note: the formulation with $i\epsilon^\pm$ regularization not equivalent to the "near-light cone" approach – the latter rotates the variables $x^\pm$ by a small angle, keeping them real, while the former one shifts the arguments slightly to the complex plane.

a regularization of the field operator by $x^+ \to (x^+ \pm i\epsilon)$ was suggested by Nakanishi and Yabuki (LMP 1977) for the purpose of "setting $x^+ = 0$

whenever one wishes". Our approach justifies the necessity to introduce small imaginary parts in both $x^{\pm}$ variables by mathematical consistency of the LF quantization, namely by the very existence of the corresponding integrals (GR), which have singularities if one starts with *both* $\epsilon^{\pm} = 0$.

A fully paralel treatment can be given for the (3+1)-dimensional theory. The corresponding field expansion

$$\phi(x) = \int\limits_0^{\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \int\limits_{-\infty}^{+\infty} \frac{d^2 k_\perp}{2\pi}$$

$$\times \Big[a(k^+, k_\perp) e^{-\frac{i}{2}k^+(x^- - i\epsilon^-) - \frac{i}{2}\frac{k_\perp^2 + \mu^2}{k^+}(x^+ - i\epsilon^+) + ik_\perp \cdot x_\perp} +$$

$$+ a^\dagger(k^+, k_\perp) e^{\frac{i}{2}k^+(x^- + i\epsilon^-) + \frac{i}{2}\frac{k_\perp^2 + \mu^2}{k^+}(x^+ + i\epsilon^+) - ik_\perp \cdot x_\perp} \Big], \tag{83}$$

where $d^2k_\perp \equiv dk^1dk^2, k_\perp \cdot x_\perp \equiv k^1x^1 + k^2x^2$, again contains the regulating terms. They are required for the existence of the integral over the k^+ variable (after performing the $d^2k_\perp$ integration) in the two-point function

$$iD^{(+)}(z) = \int_0^\infty \frac{dk^+}{4\pi k^+} \int_{-\infty}^{+\infty} \frac{d^2k_\perp}{(2\pi)^2} \times e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{k_\perp^2+\mu^2}{k^+}(z^+ - i\epsilon^+) + ik_\perp \cdot z_\perp}. \tag{84}$$

For $x^+ > 0$, the result is

$$iD^{(+)}(x) = \frac{\mu\theta(-x^2)}{4\pi^2\sqrt{-x^2}} K_1(\mu\sqrt{-x^2}) + \frac{i\mu\theta(x^2)}{8\pi\sqrt{x^2}} H_1^{(2)}(\mu\sqrt{x^2}), \tag{85}$$

where $x^2 = x^+x^- - x_\perp^2$ with the $i\epsilon^\pm$ factors implicitly present. In the space-like region, in analogy to the two-dimensional case, the direct evaluation of the two-point function in terms of LF-restricted fields agrees with the value of the two-point function (85) at $x^+ = 0$:

$$iD^{(+)}(0, x^-, x_\perp) \equiv \langle 0|\phi(0, x^-, x_\perp)\phi(0,0,0)|0\rangle = \\ = \frac{\mu}{4\pi^2\sqrt{x_\perp^2 - i\epsilon^+ x^-}} K_1(\mu\sqrt{x_\perp^2 - i\epsilon^+ x^-}). \tag{86}$$

In the previous studies, the covariant result rewritten in terms of the LF variables gave the correct expression for $x^+ = 0$ (without the $i\epsilon^+ x^-$ term, however), the direct calculation of the LF two-point function (84) reproduced this result, but the two-point function (86) calculated from the fields restricted to $x^+ = 0$ failed to yield the correct result. In other words, the problem of "the order of integration and setting $x^+ = 0$ matters" removed here

The same is true for the quantities $D(x)$, $\hat{D}^{(+)}(x)$ and $\hat{D}(x)$, which differ from their 2-dim counterparts by the obvious additional $x_\perp$ dependence or the $\delta^2(x_\perp)$ factor, for example

$$\hat{D}(0, x^-, x_\perp) = \int_0^\infty d\kappa^2 \rho(\kappa^2) \left[D^{(+)}(0, x^-, x_\perp; \kappa) - c.c. \right]$$

$$\left[\phi(x^+, x^-, x_\perp), \phi(x^+, y^-, y_\perp) \right] = -\frac{i}{4} \epsilon(z^-) \delta^{(2)}(z_\perp). \quad (87)$$

VI. LF TWO-POINT FUNCTION IN THE $x \to 0$ LIMIT

The regularized field expansion (83) also solves the **apparent failure of the LF on-shell formalism** to reproduce correctly the time-ordered two-point function

$$
\begin{aligned}
iD_F(x-y) &= \theta(x^+ - y^+)\langle 0|\phi(x)\phi(y)|0\rangle + \\
&+ \theta(y^+ - x^+)\langle 0|\phi(y)\phi(x)|0\rangle
\end{aligned} \tag{88}
$$

at $x = y$ (Mannhein, Lowdon, Brodsky)

Eeasy to see from the expression (85) which due to the behaviour of the Bessel function $K_1(x)$ (or $Y_1(x)$) for a small value of its argument $K_1(x) \sim x^{-1}$ takes the form $(x^2 < 0, x^+ > 0)$

$$
D^{(+)}(x) = \frac{i}{4\pi^2 x^2} = \frac{i}{4\pi^2} \frac{1}{(x^+ - i\epsilon^+)(x^- + i\epsilon^-) - x_\perp^2} \tag{89}
$$

and thus behaves as $(\epsilon^+\epsilon^-)^{-1}$ for $x^+ = x^- = x_\perp = 0$.

This is nothing but a regularized form of the tadpole Feynman diagram in the x-space, which in terms of the momentum-space cutoff Λ diverges as Λ^2.

In the LF version of the Feynman formalism, the expression for the 2-point function at $x = 0$ derived using the integral α-representation or from the contribution of a circle of the radius $R \to \infty$ in the complex k^--plane. The obtained result, formally mass-dependent, was however ill defined, as the corresponding integral representation

$$D^{(+)}(0) \sim \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\frac{\lambda}{\alpha} - i\mu^2\alpha - \epsilon\alpha} {}_{|\lambda \to 0} \sim \frac{1}{\lambda} \tag{90}$$

diverges for the considered case $\lambda = 0$. Presence of a non-zero ϵ does not regulate the integral.

obvious from the Eq.(89) that the LF on-shell formalism not only does not fail, it actually yields the correct result in the regularized form (89).

It is not necessary to require from the LF Hamiltonian scheme to reproduce the ill-defined form $D^{(+)}(0)$ shown in (90) because the scheme, when applied carefully, generates a mathematically superior (well-defined) form of $D^{(+)}(0)$

IV. SUMMARY AND CONCLUSIONS

- the 2D massless fields can be correctly quantized including the gauge field $A^{\mu}(x)$
- LF scalar, fermion and gauge fields contain a zero-mode component in 2D; what about 4D?

- the LF restriction of the scalar-field two-point function is well defined and mass-dependent

- its value at coinciding points correctly reproduced in the onshell Hamiltonian formulation

LF field theory has its subtleties but it is a consistent version of QFT

- What next?

- LF Federbush model, priority: operator solution of the LF Schwinger model

- massive versions of the solvable models: Rothe-Stamatescu, massive vector boson, Thirring ...

- LF QED$(3+1)$ in the covariant gauge

- dynamical LF zero modes in D$=(3+1)$, presumably exist only for massless fields

- apply LF Hamiltonian perturbation theory to LF Feynman amplitudes with "treacherous points" / LF singularities