LF field theory: vacuum diagrams, massless fields, solvable models, zero modes, two-point functions

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ABSTRACT: I will give an overview of selected results in light-front (LF) field theory obtained within the last two decades. First, I will demonstrate advantages of the (not manifestly-covariant) **LF Hamiltonian**

perturbation theory by calculating **forward-scattering amplitude** in two-dimensional self-interacting scalar model in a finite-volume treatment (DLCQ method). The corresponding amplitudes converges to the correct continuum limit **without a need for a LF zero mode**. In a similar fashion, **non-vanishing LF vacuum amplitudes** will be shown to exist in the LF theory despite naive kinematical arguments that forbid the vacuum bubbles. In the second lecture, I will formulate a consistent quantization scheme for **two-dimensional massless LF fields**. Based on that, **non-perturbative (operator) solutions** of a few two-dimensional relativistic models (derivative-coupling, Thirring, Thirring-Wess) will be given in the LF formulation. For comparison, a systematic Hamiltonian treatment of the **Thirring model within the conventional field theory** will be presented including the **true physical ground state** of the model with the non-trivial structure. In the third lecture, I shall start with quantization of the **two-dimensional LF abelian gauge field** as a massless limit of the massive vector field in the covariant (Feynman) gauge and will show presence of an infinite set of **dynamical LF zero modes**. Then a few more formal/structural aspects of the LF QFT will be addressed, in particular consistency of **boundary values of the two-point functions** in two and four dimensions, and the ability of the LF Hamiltonian approach to yield the correct result in agreement with the covariant Feynman approach. An important element in this demonstration is the field expansions with small imaginary parts in the plane-wave factors. Finally, an outline of future work along the presented topics is given.

MY CV

- 1980, diploma thesis, theoretical physics, Comenius University, Bratislava, Czechoslovakia
- 1987, PhD thesis, Institute of Physics, Slovak Academy of Sciences,

Bratislava, pion elm form factor

- 1987-1990, postdoc, Laboratory of Theoretical Physics JINR, Dubna, USSR, interest in Pauli&Brodsky DLCQ/LF field theory
- 1991 - pending, senior scientific worker, Institute of Physics Bratislava
- 1991-1992, postdoc, Max-Planc Institute of Kernphysik, Heidelberg, Germany, group of Prof. H.-C. Pauli
- 1996-2000, NSF grant with Prof. J. Vary, Iowa State University, Ames
- 2004 -2005, NATO fellowship, work with Prof P. Grange, Montpellier University, France
- 2010-2020, senior researcher, Laboratory of Theoretical Physics, JINR Dubna, Russia

Selected results over last 15 years

A few remarks concerning the Motivation - Purpose - Conceptual basis

- To demonstrate advantages of (manifestly-non covariant) Hamiltonian LF perturbation theory (PT)
- Delicate role/status/real existence of the LF zero modes
- In perturbation theory, ZMs not needed, Feynman amplitudes rewritten in terms of the LF variables do not fully coincide with genuine LF PT formalism/mechanisms
- nicely illustrated with the example of LF vacuum amplitudes (bubbles)
- LF vacuum - too simple? - where are the vacuum phenomena, degeneracy, symmetry breaking?

- Solvable models: toy (unrealistic) theories in 2D, non-perturbative (operator) solutions of field equations possible, suitable for comparison between the LF and conventional "ET" form of the theory

- LF "failures" or apparent inconsistencies: try positive approach, solutions may differ from the ET patterns

Light front (LF) field theory: an independent form of QFT

different parametrization of space-time $\Rightarrow$ different structure of field equations, field variables (dynamical vs. constrained) ...

Dirac RMP 1949:

three forms of the Hamiltonian relativistic dynamics

front form the most efficient one, only 3 dynamical Poincaré generators, physical vacuum obtained kinematically (no need to solve the dynamics as in the conventional ("instant" or space-like (SL) form), positivity of the (kinematical) quantity - the LF momentum p^+ in addition to the (LF) energy P^-.

Here $p^{\pm} = p^0 \pm p^3, p^- = \frac{p_\perp^2 + m^2}{p^+}$ - no square-root ambiguity, $\partial_\mu \partial^\mu = \partial_+ \partial_- - \partial_\perp^2 \Rightarrow$ different structure of field equation, smaller number of dynamical dofs (constraints), etc.

creation and annihilation operators well defined also outside the quantization surface $x^+ = 0$, not true for SL form outside $t = 0$ (BD)

Quantum field theory formulated in terms of light-front variables:

needs new/different intuition and careful mathematics

Example: massless fields in 2 dimensions: seemingly hard to initialize, quantization appeared obscure, ad hoc constructions... long-lasting struggle with the Schwinger model (!)

In fact they emerge as the massless limit of massive fields (scalar, fermion...). Based on the massive 2-point functions, change of variables for some components. A consistent scheme, correct consequences (solvable models, bosonization, conformal field theory...)

Another problem appeared to be paradoxically related to the most celebrated property of the LF quantization - vacuum simplicity

Fock vacuum is an exact eigenstate with lowest energy for the full interacting Hamiltonian

positivity of the LF momentum p^+ together with its conservation implies that the ground state of any dynamical model cannot contain quanta carrying $p^+ \neq 0$. Only a tiny subset of all field modes, namely those carrying $p^+ = 0$ - the LF zero modes (ZM) - can contribute:

VACUUM EFFECTS, STRUCTURE?

I. VACUUM DIAGRAMS IN LIGHT-FRONT FIELD THEORY

ABSTRACT:

We demonstrate that vacuum diagrams in light front field theory are non-zero, contrary to the prevailing opinion. Using the light-front Hamiltonian (time-ordered) perturbation theory, the vacuum amplitudes in self-interacting scalar $\lambda\phi^3$ and $\lambda\phi^4$ models are obtained as $p = 0$ limit of the associated self-energy diagrams, where p is the external momentum. They behave as $C\lambda^2\mu^{-2}$ in D=2, with μ being the scalar-field mass, or diverge in D=4, in agreement with the usual "equal-time" form of field theory, and with the same value of the constant C. The simplest case of the vacuum bubble with two internal lines is analyzed in detail. It is shown that, surprisingly, the light-front diagrams are nonvanishing not due to the zero-mode contribution. This is made explicit using the DLCQ method - the discretized (finite-volume) version of the theory, where the light-front zero modes are manifestly absent, but the vacuum amplitudes still converge

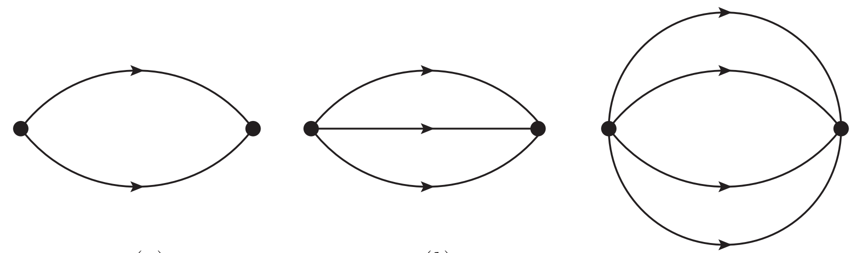
to their continuum-theory values with the increasing "harmonic resolution" K. Finally, a few open (and a bit puzzling) questions will be discussed.

I. INTRODUCTION

How does LF theory describe vacuum phenomena? Is the LF dynamics equivalent to the SL one? Can it predict something new?

Prevailing opinion (Brodsky, Burkhard...): LF vacuum always "trivial" (empty state), in particular: vacuum bubbles do not exist in LF perturbation theory, cosmological consequences (Brodsky and Schrock) if true would mean LF theory is not equivalent to the SL theory

In 2018, **J. Collins** has pointed out this controversy along with a corrected treatment for the simplest LF vacuum loop with 2 internal lines, identified a mathematical difficuty



Vacuum bubbles for ϕ^3 and ϕ^4 models

The equivalence issue realized and studied already in the pioneering papers on LF perturbative S-matrix by Cheng and Ma (1969) and by T.-M. Yan (1973) includig the vacuum problem at the perturbation theory level

Method: covariant Feynman amplitudes (integrals) rewritten in terms of LF variables, **the delicate step:** to perform the integration in p^- variable, since the propagators in 2D behave as $(k^+k^- - m^2 + i\epsilon)^{-1}$ instead of $(k_0^2 - k_1^2 - m^2 + i\epsilon)^{-1}$ - convergence

T.-M. Yan, PRD 7, 1780 (1973): $I = \int d^4p \frac{1}{(p^2-\mu^2+i\epsilon)^3} = \frac{\pi^2}{2i\mu^2}$.
Here $d^4p = dp^0 dp^1 dp^2 dp^3$ and $p^0 \to idp^4$. In LF variables,

$$I = \int dp^+ dp^- d^2p_\perp \frac{1}{(p^+p^- - p_\perp^2 - \mu^2 + i\epsilon)^3} = -\frac{\pi}{4}\int dp^+ dp^- \frac{1}{(p^+p^- - \mu^2 + i\epsilon)^2}. \qquad (1)$$

A double pole at $p^- = \frac{\mu^2 - i\epsilon}{p^+}$, at infinity for $p^+ = 0$, a careful treatment needed:

$$I = -\frac{\pi}{4}\int_{-\infty}^{+\infty} dp^+ \lim_{\Lambda\to\infty}\int_{-\Lambda}^{+\Lambda} dp^- \frac{1}{(p^+p^- - \mu^2 + i\epsilon)^2} =$$

$$= \frac{\pi}{4}\int_{-\infty}^{+\infty} \frac{dp^+}{p^+} \lim_{\Lambda\to\infty}\Big(\frac{1}{p^+\Lambda - \mu^2 + i\epsilon} - \frac{1}{-p^+\Lambda - \mu^2 + i\epsilon}\Big). \qquad (2)$$

Using the identity

$$\frac{1}{p^+}\Big(\frac{1}{p^+\Lambda-\mu^2+i\epsilon}-\frac{1}{-p^+\Lambda-\mu^2+i\epsilon}\Big)=\frac{1}{\mu^2}\Big(\frac{\Lambda}{p^+\Lambda-\mu^2+i\epsilon}-\frac{\Lambda}{p^+\Lambda+\mu^2-i\epsilon}\Big), \tag{3}$$

for $\Lambda \to \infty$, one gets

$$I=\frac{\pi}{4\mu^2}\int_{-\infty}^{+\infty} dp^+\Big(\frac{1}{p^++i\epsilon}-\frac{1}{p^+-i\epsilon}\Big)=\frac{\pi}{4\mu^2}\int_{-\infty}^{+\infty} dp^+\big[-2i\pi\delta(p^+)\big]=\frac{\pi^2}{2i\mu^2}. \tag{4}$$

Same result with the exponential α-representation $iD^{-1}=\int_0^\infty d\alpha e^{i\alpha(D+i\epsilon)}$.

Chang and Ma different method for a vacuum bubble with 3 internal lines

$$V=\int dp^+dp^-\frac{1}{p^+p^--\mu^2+i\epsilon}\Sigma(p^+p^-), \tag{5}$$

where $\Sigma(p^2)$ represented as

$$\Sigma(p^2) = \int d\lambda F(\lambda) e^{i\lambda p^2}, \;\; F(\lambda) = \int\limits_0^{+\infty} d\alpha_1 d\alpha_2 \delta(\lambda(\alpha_1{+}\alpha_2){-}\alpha_1\alpha_2) e^{-i\mu^2\alpha_1\alpha_2/\lambda}, \tag{6}$$

where the above α-representation used here and also in (5). Insert Σ of (6) into (5):

$$V = \int dp^+ dp^- \big[- i \int\limits_0^{+\infty} d\alpha d\lambda F(\lambda) e^{ip^2(\lambda+\alpha) - i\mu^2\alpha} \big] =$$

$$= \int dp^+ \Big[- 2\pi i \int_0^{+\infty} d\alpha d\lambda F(\lambda) (\lambda+\alpha)^{-1} e^{-i\mu^2\alpha} \Big] \delta(p^+). \tag{7}$$

Non-zero result, but no explicit formula given.

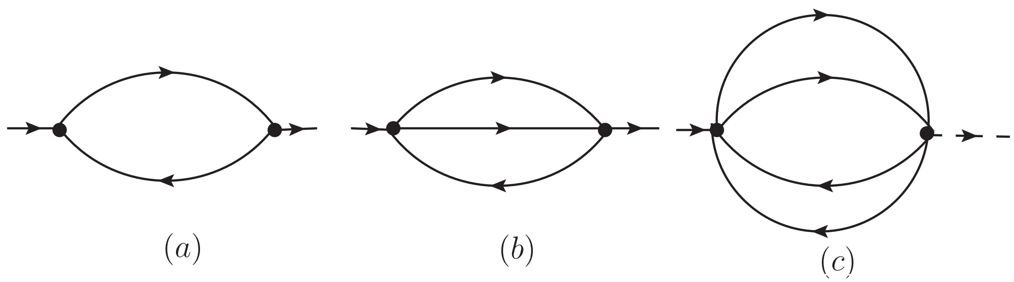


Figure 2. Self-energy diagrams for for ϕ^3, ϕ^4 and ϕ^5 models

HERE: generalization of Collin's analysis to loops with more internal lines, using the analyticity argument, both continuum and finite-volume formulation (DLCQ), complete agreement with covariant Feynman results

"HISTORICAL" DETOUR

A. Harindranath, K. Martinovic, J. P. Vary, Perturbative S-matrix in discretized light cone quantization of two-dimensional ϕ^4 theory, Phys. Lett. B 536 (2002) 250 illustrates (i) simplicity of "old-fashioned" LF perturbation theory, starting from covariant Feynman amplitudes may be misleading (ii) works in a finite volume with correct continuum limit (iii) no ZMs needed

the paper was response to the work M. Taniguchi, S. Uehara, S. Yamada, K. Yamawaki, Does DLCQ S-matrix have a covariant continuum limit?, Mod. Phys. Lett. A16 (2001) 2177

CLAIM: NO, IT DOES NOT, at least for processes with $p^+ = 0$ exchange

PROBLEM: they considered 2-dimensional integrals $\int dk^- dk^+ ..., k^+$ discretized, changes of variables $\Rightarrow$ forward scattering amplitude vanishes in the continuum limit – **wrong**

our approach actually gave non-vanishing vacuum amplitudes, correct value $-\frac{\lambda^2}{8\pi\mu^2}$

II. THE FORMALISM AND SIMPLE EXAMPLES

The basic formula for the S-matrix in the "old-fashioned", Hamiltonian, LF-time ordered, non-manifestly covariant PT (it avoids the k^- integration in a natural way, also: energy denominators instead of covariant propagators)

Like in the covariant treatment, one starts, in the interacting representation, from

$$i\frac{\partial\Phi(t)}{\partial t} = H_I(t)\Phi(t), \Phi(t) = S(t,t_0)\Phi(t_0) \Rightarrow i\frac{\partial S(t,t_0)}{\partial t} = H_I(t)S(t,t_0). \tag{8}$$

Try to find the solution as $\sum_n g^n S_n(t,t_0)$. Inserting to the second eq. above:

$$i\frac{\partial S_0(t,t_0)}{\partial t} = 0, i\frac{\partial S_1(t,t_0)}{\partial t} = H_I(t)S_0, ...i\frac{\partial S_n(t,t_0)}{\partial t} = H_I(t)S_{n-1} \tag{9}$$

Integrating

$$S_1(t_1, t_0) = -i \int_{t_0}^{t_1} dt_2 H_I(t_2), \tag{10}$$

etc.

With $V = P^-_{int}$, $\quad V(x^+) = e^{\frac{i}{2}P_0^- x^+} V(0) e^{-\frac{i}{2}P_0^- x^+}$, we have

$$S_{fi} = \delta_{fi} - \frac{i}{2} \int_{-\infty}^{+\infty} dx^+ \langle \phi_f | V(x^+) | \phi_i \rangle -$$

$$-\frac{1}{4} \int_{-\infty}^{+\infty} dx_1^+ \langle \phi_f | V(x_1^+) | \phi_n \rangle \int_{-\infty}^{x_1^+} dx_2^+ \langle \phi_n | V(x_2^+) | \phi_i \rangle + \ldots , \tag{11}$$

The T and M matrices are defined after extracting kinematical factors:

$$S_{fi} = \delta_{fi} - 2\pi i \delta(p_i^- - p_f^-) T_{fi}, \quad T_{fi} = \frac{1}{\sqrt{p_f^+ p_i^+}} \delta(p_f^+ - p_i^+) M_{fi} \tag{12}$$

A complete set of states was inserted in (11):

$$\hat{1} = \sum_n |\phi_n\rangle\langle\phi_n| = |0\rangle\langle 0| + \int_0^{+\infty} dl_1^+ a^\dagger(l_1^+)|0\rangle\langle 0|a(l_1^+) +$$
$$+ \int_0^{+\infty} dl_1^+ \int_0^{+\infty} dl_2^+ a^\dagger(l_2^+) a^\dagger(l_1^+)|0\rangle\langle 0|a(l_1^+) a(l_2^+) + \ldots . \tag{13}$$

We shall work with $\lambda\phi^3$ and $\lambda\phi^4$ models in 2D, for which

$$P^-_{int} = \frac{\lambda}{3!} 3 \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi p^+}} \int_0^{+\infty} \frac{dq^+}{\sqrt{4\pi q^+}} 2\pi\delta(p^+ + k^+ - q^+)$$
$$\times \left\{ a^\dagger(q^+) a(k^+) a(p^+) + a^\dagger(p^+) a^\dagger(k^+) a(q^+) \right\} \qquad (14)$$

$$P_{int}^- = V_1 + V_2 + V_3 =$$

$$= \frac{\lambda}{4!} \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi p^+}} \int_0^{+\infty} \frac{dq^+}{\sqrt{4\pi q^+}} \int_0^{+\infty} \frac{dr^+}{\sqrt{4\pi r^+}} 8\pi$$

$$\times \Big\{ \Big[a^\dagger(k^+) a^\dagger(p^+) a^\dagger(q^+) a(r^+) + a^\dagger(r^+) a(p^+) a(q^+) a(k^+) \Big] \delta(k^+ + p^+ + q^+ - r^+)$$

$$+ \frac{3}{2} a^\dagger(k^+) a^\dagger(p^+) a(q^+) a(r^+) \delta(k^+ + p^+ - q^+ - r^+) \Big\}. \quad (15)$$

The rules of the LF perturbation theory imply that the vacuum amplitudes (bubbles) vanish (or rather are mathematically ill-defined) (Yan 1973) as the corresponding integrals contain the delta function $\delta(p_1^+ + p_2^+ ... + p_n^+)$ (momentum conservation) which can be satisfied only if all of them vanish, leading to singular integrands. The simplest example: **LF tadpole**

$$a^\dagger(k_1^+) a(k_2^+) a^\dagger(k_3^+) a(k_4^+) = \delta(k_2^+ - k_3^+) a^\dagger(k_1^+) a(k_4^+) + no \equiv V_T + no, \quad (16)$$

- it arises in the process of normal-ordering the Hamiltonian

$$S_{fi}^{(1)} = -\frac{i}{2} \int_{-\infty}^{+\infty} dx^+ \langle 0 | a(p_f^+) e^{\frac{i}{2} p_f^- x^+} V_T e^{-\frac{i}{2} p_i^- x^+} a^\dagger(p_i^+) | 0 \rangle \quad (17)$$

$$\Rightarrow M_T = \frac{\lambda}{8\pi} \int_0^{+\infty} \frac{dk^+}{k^+} \to \frac{\lambda}{8\pi} \int_\epsilon^\Lambda \frac{dk^+}{k^+} = \frac{\lambda}{8\pi} \int_{\frac{\mu^2}{\Lambda}}^\Lambda \frac{dk^-}{k^-} = \frac{\lambda}{8\pi} \log \frac{\Lambda^2}{\mu^2}. \quad (18)$$

change of variable $k^+ \to \frac{\mu^2}{k^+}$ performed, no need to hunt for poles at infinity/"Ligterink method" (complicated), Mishchenko et al, PRD 2005

- A. Harindranath, L. Martinovic and J. P. Vary, PRD 64, 105016 (2001):

IMF, near-LC and LFPT loop diagrams (self-energy and scattering):

comparison

in particular, one-loop self-energy in $\lambda\phi^3 (3+1)$ toy model

$$\Sigma(p^2) = \frac{\lambda^2}{4(2\pi)^3} \int_0^1 dx \int d^2 q_\perp \frac{1}{p^2 x(1-x) - (q_\perp)^2 - \mu^2 + i\epsilon}. \tag{19}$$

Reducing to 1+1 dim and setting $p = 0$ (=vacuum bubble, J. Collin's case), we have

$$V \equiv \Sigma(0) = \frac{\lambda^2}{8\pi} \int_0^1 dx \frac{1}{-\mu^2 + i\epsilon} = -\frac{1}{8\pi} \frac{\lambda^2}{\mu^2}. \tag{20}$$

We did not realize this connection at that time

Simple case - analytic formula for $s \equiv p^2 \neq 0$:

$$V = \frac{\lambda^2}{8\pi} \int_0^1 \frac{dx}{sx(1-x) - \mu^2 + i\epsilon} = -4 \frac{\arctan\sqrt{\frac{s}{4\mu^2 - s}}}{\sqrt{4\mu^2 s(1-s)}} \frac{\lambda^2}{8\pi}. \tag{21}$$

Undefined for $s = 0$, L'Hospital yields the correct value $\sim -1/\mu^2$.

Vacuum amplitudes in the SL form: bubble in ϕ^3 toy model

The corresponding Feynman rules lead to the double two-dimensional integral expression

$$V_3(\mu) = N\lambda^2 \int d^2k_1 \int d^2k_2 \; G(k_1)G(k_2)G(k_1+k_2), \tag{22}$$

$$G(k) = \frac{i}{k^2 - \mu^2 + i\epsilon}, \qquad N = \frac{1}{3!}\frac{1}{i(2\pi)^4}. \tag{23}$$

The coefficient $1/3!$ is the symmetry factor. Can be evaluated in a few ways: by using the Feynman parameters, by means of α-representation or via more sophisticated mathematical methods (Mellin-Barnes representation for powers of massive propagators (Davydychev and Tausk, NPB (1993), PRD (1996)) - the same result

$$V_3(\mu) = -\frac{i\lambda^2}{\mu^2}\pi^2 NC, \quad C = 2.343908..., \tag{24}$$

The constant C has a particular representation in each of the methods

The first method: combine the propagators into one denominator by means of the auxiliary integrals in terms of the Feynman parameters x_i, then go over to Euclidean space and calculate the integrals in p and q

variables. The result is the the double-integral representation

$$C = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{1}{D(x_1, x_2)}, \tag{25}$$

$$D(x_1, x_2) = x_1(1 - x_1) + x_2(1 - x_2) - x_1 x_2.$$

The first integral can be calculated analytically in terms of $arctan$ function and square-roots of polynomials, the numerical evaluation of the second integral then yields the above value of C.

If we consider the self-energy diagram instead of the vacuum bubble ($G(k_1+k_2)$ replaced by $G(p-k_1-k_2)$ in Eq.(22)), the analogous calculation yields

$$\Sigma_4(p^2) = N\lambda^2 \int\limits_0^1 dx_1 \int\limits_0^{1-x_1} dx_2 \frac{1}{A(x_1,x_2)p^2 - D(x_1,x_2)\mu^2},$$
$$A(x_1,x_2) = x_1x_2(1-x_1-x_2). \quad (26)$$

matches the LF result for $p=0$

III. LIGHT-FRONT CALCULATION IN CONTINUUM

The result (24) obtained in a very simple way also in the LFPT, contrary to the the general belief

Naively, the LFPT rules yield ill-defined expression (Yan 1973)

$$\tilde{V} \sim \int_0^\infty \frac{dp_1^+}{p_1^+} \int_0^\infty \frac{dp_2^+}{p_2^+} \int_0^\infty \frac{dp_3^+}{p_3^+} \frac{\delta(p_1^+ + p_2^+ + p_3^+)}{(-\mu^2)[\frac{1}{p_1^+} + \frac{1}{p_2^+} + \frac{1}{p_3^+}]}. \tag{27}$$

This essentially expresses the fact that since the incoming momentum is zero, the conservation of the LF momentum p^+ requires that each of three internal lines must also carry vanishing LF momentum, **BUT** there is no such a mode!

THE CORRECT METHOD: start with the (self-energy) graph with nonvanishing external momentum, write down the corresponding LF amplitude and evaluate its value at $p = 0$ (after going over to relative LF momenta – covariant form):

the vacuum loop emerges simply as the limit of the associated self-energy graph (of the "n+1" theory) for vanishing external momentum:

$$\Sigma_4(p) = \tilde{N}\lambda^2 \int_0^{p^+} \frac{dk^+}{k^+} \int_0^{p^+-k^+} \frac{dl^+}{l^+(p^+ - k^+ - l^+)} \frac{1}{p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{l^+} - \frac{\mu^2}{p^+-k^+-l^+} + i\epsilon}. \quad (28)$$

Introducing the dimensionless variables $x = \frac{k^+}{p^+}$, $y = \frac{l^+}{p^+}$, $\Sigma(p)$ becomes

$$\Sigma_4(p) = \tilde{N}\lambda^2 \int_0^{1} \frac{dx}{x} \int_0^{1-x} \frac{dy}{y(1-x-y)} \frac{1}{\left[p^2 - \frac{\mu^2}{x} - \frac{\mu^2}{y} - \frac{\mu^2}{1-x-y}\right]}, \ \tilde{N} = \frac{1}{3!}\frac{1}{4\pi)^2}. \quad (29)$$

Now we can set $p = 0$. This expression replaces the incorrect Eq.(27).

The integral over the variable y can be performed explicitly, yielding

$$F(x) = \frac{1}{\mu^2} \frac{4x}{\sqrt{3x^2 - 2x - 1}} \arctan \frac{1-x}{\sqrt{3x^2 - 2x - 1}}. \tag{30}$$

The numerical computation of the second integral gives

$$\Sigma_4(0) \equiv V_3(\mu) = -\frac{\lambda^2}{\mu^2} \tilde{N} C, \quad C = 2.343908\ldots, \tag{31}$$

in the complete agreement with the space-like result (24). The overall situation very simple - multiplying out the terms in the denominator of Eq.(29) for $p = 0$: the corresponding double integral is precisely equal the representation of the constant C in Eq.(26). The LF and SL schemes match at this point. The only difference:

in the SL theory one can start directly with the vacuum diagram while in the LF case the latter emerges as the the limit of the associated self-energy diagram for vanishing external momentum

change to relative variables makes the integrand to depend on p^2, i.e. symmetrically on both p^+ and p^-.

$\Sigma_4(p^2)$ is also the same in the both schemes, the difference: the LF scheme needs for that just two steps, the conventional Feynman procedure by an order of magnitude longer

THE SIMPLEST DIAGRAM WITH TWO INTERNAL LINES

The above result can be confirmed in a different way. Consider for simplicity the one-loop self energy diagram in the $\lambda\phi^3$ theory (Fig. 2). The corresponding Feynman amplitude is

$$-i\Sigma_3(p^2) = \frac{1}{2}\frac{(-i\lambda)^2}{(2\pi)^2}\int d^2k\ G(k)G(p-k), \tag{32}$$

The vacuum bubble ($p = 0, \lambda = 1$) rewritten in terms of the LF variables:

$$V_2(\mu) = \frac{i}{16\pi^2} \int\limits_{-\infty}^{+\infty} dk^+ \int\limits_{-\infty}^{+\infty} dk^- \frac{1}{(k^+k^- - \mu^2 + i\epsilon)^2}. \tag{33}$$

To correctly evaluate the integral over k^-, impose a cutoff Λ [**?**], leading to ($c = -i/16\pi^2$)

$$V_2(\mu) = c \int\limits_{-\infty}^{+\infty} \frac{dk^+}{k^+} \Big[\frac{1}{\Lambda k^+ - \mu^2 + i\epsilon} - \frac{1}{-\Lambda k^+ - \mu^2 + i\epsilon} \Big].$$

Utilizing a suitable identity (see above), one finds for $\Lambda \to \infty$

$$V_2(\mu) = \frac{c}{\mu^2} \int_{-\infty}^{+\infty} dk^+ \Big[\frac{1}{k^+ + i\epsilon} - \frac{1}{k^+ - i\epsilon} \Big] =$$

$$= \frac{c}{\mu^2}(-2\pi i) \int_{-\infty}^{+\infty} dk^+ \delta(k^+) = -\frac{1}{8\pi} \frac{1}{\mu^2}. \qquad (34)$$

agreement with the result obtained in the usual SL calculation

This simplest diagram sheds light upon the mechanism at work in the genuine LF case

The LFPT formula for the self-energy Σ_3 (32) is

$$\Sigma_3(p) = \frac{\lambda^2}{8\pi} \int\limits_0^{p^+} \frac{dk^+}{k^+(p^+ - k^+)} \frac{1}{p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{p^+ - k^+} + i\epsilon}. \tag{35}$$

introduce variable $x = k^+/p^+$, the denominator transformed to the form $x(1-x)p^2 - \mu^2 + i\epsilon$; for $p = 0$ (34) reproduced

Alternatively, work directly with the form (35). Take $p^+ = p^- = \eta$ for simplicity, we have ($\lambda = 1$ henceforth)

$$\Sigma_3(\eta) = \frac{1}{8\pi} \int\limits_0^{\eta} \frac{dk^+}{k^+(\eta - k^+)} \frac{1}{\eta - \frac{\mu^2}{k^+} - \frac{\mu^2}{\eta - k^+} + i\epsilon}. \tag{36}$$

The integral evaluated exactly:

$$\Sigma_3(\eta) = -\frac{1}{4\pi}\Big(G(\eta) - G(0)\Big),\ G(k) = \frac{\arctan\left(\frac{2k-\eta}{\sqrt{4\mu^2-k^2}}\right)}{\eta\sqrt{4\mu^2-\eta^2}}. \quad (37)$$

The expansion for infinitesimal η gives

$$\Sigma_3(\eta) = -\frac{1}{8\pi}\frac{1}{\mu^2}\Big[1 + \frac{\eta^2}{4\mu^2} + \mathcal{O}(\eta^4)\Big]. \quad (38)$$

The correct result recovered for $\eta = 0$.

The reason is simple: the integrand in (36) is $\eta^{-1}[k^+(\eta - k^+) - \mu^2]^{-1}$. For very small η the expression in the brackets almost a constant very close to $(-\mu^2)$ at the interval $(0, \eta)$, while the diverging η^{-1} factor is canceled by the length η of the integration domain. However, setting $\eta = 0$ from very beginning as in (27) yields the ill-defined result

FINITE VOLUME (DLCQ) CALCULATION

Remarkably, the same result obtained in the discretized (finite-volume) treatment with (anti-)periodic boundary conditions (BC). In both cases, the field mode carrying $k^+ = 0$ is manifestly absent.

The corresponding field expansion at $x^+ = 0$ is

$$\phi(0, x^-) = \frac{1}{\sqrt{2L}} \sum_n^{\infty} \frac{1}{\sqrt{k_n^+}} \left[a_n e^{-ik_n^+ x^-} + a_n^\dagger e^{ik_n^+ x^-} \right], \qquad (39)$$

where $k_n^+ = 2\pi n/L$ and L is the lenght of the finite interval. The index n runs over half-integers for antiperiodic boundary conditions and over integers for periodic BC, with $n = 0$ excluded (the field equation in the ZM sector $\mu^2 \phi_0 = 0$ requires the field mode $\phi_0 \equiv \phi(k^+ = 0)$ to vanish for $\mu \neq 0$.) The DLCQ analog of the $\Sigma_3(p)$ amplitude is

$$\Sigma_3(p) = \mathcal{N} \sum_{k^+}^{p^+} \frac{1}{k^+(p^+ - k^+)} \frac{1}{\left[p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{p^+ - k^+}\right]}, \tag{40}$$

$$p^+ = \frac{2\pi}{L} K, \;\; k^+ \equiv k_n^+ = \frac{2\pi}{L} n, \;\; n = 1, 2, \ldots, K - 1. \tag{41}$$

Normalization constant $\mathcal{N}$. For $p = 0$ and PBC, $\Sigma_3(0) = V_2(\mu)$ is

$$
\begin{aligned}
V_2(\mu) &= -\frac{1}{8\pi\mu^2} S, \\
S &= \sum_{n=1}^{K-1} \frac{1}{n(K-n)} \frac{1}{\left[\frac{1}{n} + \frac{1}{K-n}\right]} = \frac{K-1}{K}.
\end{aligned} \tag{42}
$$

K plays the role of η^{-1} For $K \to \infty$, S converges to the continuum value 1. The same result for the antiperiodic BC

In Table I, the smooth approach of the self-energy value to the vacuum-loop value as $p \to 0$ is shown.

Table 1: Smooth approach of the one-loop self-energy amplitude of the $\lambda\phi^3$ model to its value at $p = 0$ $(K = 512)$

p^2/μ^2	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	0
C	1.50902	1.09320	1.00667	0.99890	0.99813	0.99805

The DLCQ analog of the $\Sigma_4(p)$ amplitude is

$$\Sigma_4(p) = -\lambda^2 N \sum_{q^+}^{p^+} \sum_{k^+}^{p^+-q^+} \frac{1}{k^+ q^+ (p^+ - k^+ - q^+)\left[p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{q^+} - \frac{\mu^2}{p^+-k^+-q^+}\right]}. \tag{43}$$

For $p = 0$ and with $k^+ \to \frac{2\pi}{L} m$, etc.:

$$\Sigma_4(0) = V_3(\mu^2) = -\frac{1}{\mu^2} \sum_{m=1}^{K-2} \frac{1}{m} \sum_{n=1}^{K-m-1} \frac{1}{n(K-m-n)\left[\frac{1}{m} + \frac{1}{n} + \frac{1}{K-m-n}\right]}. \tag{44}$$

Numerical values:

$$\begin{array}{ccccc} K = 32 & K = 64 & K = 128 & K = 512 & K = 2048 \\ V = 1.921 & V = 2.099 & V = 2.205 & V = 2.301 & V = 2.331 \end{array} \tag{45}$$

Smooth approach to $p = 0$ ($K = 512$):

$$\begin{array}{cccc} p^2/\mu^2 = 10^{-2} & p^2 = 10^{-4} & p^2 = 10^{-6} & p^2 = 0 \\ V = 3.267 & V = 2.307 & V = 2.301 & V = 2.302 \end{array} \tag{46}$$

Convergence for the ϕ^4 loop slower, but reliable:

$$V_4 = -\frac{1}{\mu^2} \sum_{l=1}^{K-3} \frac{1}{l} \sum_{m=1}^{K-l-2} \frac{1}{m} \sum_{n=1}^{K-l-l-1} \frac{1}{n(K-l-m-n)\left[\frac{1}{l}+\frac{1}{m}+\frac{1}{n}+\frac{1}{K-l-m-n}\right]}. \quad (47)$$

$V_4 = 6.798, \quad 7.795, \quad 7.967$ for $K = 128, \quad 512, \quad 800$, approaching the continuum value $V_4 = 8.414....$

QUESTIONS

- is the "limiting picture" the only explanation? Note: p^+ only positive, only 1 mode propagating in vacuum loop, but it does not exist, dilemma?
- P^+ conservation violated? For arbitrarily small p manifestly not, just in one point? What could be a different interpretation of $\Sigma(0)$?

- "LF Feynman" $=$ LF theory?

- Correction to the LF vacuum energy?

CONCLUSIONS

- vacuum diagrams in the $\phi^3(1+1)$ and $\phi^4(1+1)$ models obtained as $p = 0$ (external momentum) limit of the corresponding self-energy diagrams

- works also in a final volume with (A)PBC (DLCQ) $\Rightarrow$ not effect of the zero modes

- generalization to e.g. Yukawa theory and to (3+1)-dimensional case straightforward

- expected to work also for the generalized tadpoles - to be checked

Simple tadpole and a generalized tadpole in ϕ^4 model