Massless light-front fields and solvable models

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ABSTRACT: A consistent quantization of two-dimensional massless light-front (LF) fields is formulated. They are obtained as the massless limit of the corresponding massive fields and reproduce the massless limit of the massive two-point functions. The massless LF fields can be used to derive the LF version of bosonization. They are the essential building blocks in the non-perturbative (operator) solutions of the LF version of the

Thirring and Thirring-Wess models. Before presenting these LF operator solutions, a comparison between the conventional (SL) and LF versions of the derivative coupling model will be given, and the full Hamiltonian treatment of the SL Thirring model will be described. It includes the previously neglected true physical vacuum state, obtained here by means of a Bogoliubov transformation in the coherent-state form. Bosonization of the vector current is a necessary condition for that, because it transforms the four-fermion interaction term to the bilinear form.

I. INTRODUCTION

The light front (LF) form of field theory has been praised for its potential for decades (pioneered by Dirac in 1949)

essence: QFT with different choice of the space-time and field variables

Distinguished features:

- minimal number (3) of dynamical Poincaré generators
- status of the vacuum state: Fock vacuum is (almost) the true ground state (lowest-energy eigenstate of the FULL Hamiltonian) – due to positivity and conservation of the LF momentum p^+
- consistent Fock expansion of the bound states, amplitudes with direct probabilistic interpretaion á la QM

- reduction of the number dynamical field variables, constrained components (this also technically complicates the theory)

Some doubts still present: how the LF scheme can cope with the vacuum structure, condensates, symmetry breaking...

Our overall goal two-fold: test the LF scheme at the level of solvable models in D$= 1+1$ + compare with the corresponding results within the conventional (space-like, SL) theory (massless fields often involved):

- what is the relation between the two schemes?

- can the LF theory with its drastically simplified vacuum structure generate the same predictions as the SL form?

2D massless LF fields not understood for decades attempts essentially failed - the Schwinger model, e.g.

HERE:

- a consistent quantization scheme for massless LF scalar and fermion fields derived
- application to bosonization: simple and consistent
- application to conformal symmetry: correct results generated (correlation functions, quantum Virasoro algebra)

II. MASSLESS LIGHT FRONT FIELDS IN D=1+1

LF notation: $x^\mu = (x^+, x^-) = (x^0 + x^1, x^0 - x^1)$

the momentum k^μ

$$k^\mu = (k^+, k^-), \quad \partial_\pm = \frac{\partial}{\partial x^\pm}, \quad \hat{k} \cdot x = \frac{1}{2}k^+ x^- + \frac{1}{2}\hat{k}^- x^+, \quad k^2 = \mu^2 \Rightarrow \hat{k}^- = \frac{\mu^2}{k^+}. \tag{1}$$

$\hat{k}^-$ is the on-shell LF energy. No sign ambiguity analogous to $E(k^1) = \pm\sqrt{(k^1)^2 + \mu^2}$ of the conventional theory, both k^+, k^- can be taken positive. Quantization of the massless LF fields: start from the massive ones

II.1 Massless LF scalar field

The covariant Lagrangian density + the corresponding field equation

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2, \quad \left(\partial_\mu\partial^\mu + \mu^2\right)\phi(x) = 0, \tag{2}$$

takes in terms of the LF variables the form

$$\mathcal{L} = 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2, \quad \left(4\partial_+\partial_- + \mu^2\right)\phi(x) = 0. \tag{3}$$

The corresponding conjugate momentum and the field time derivative

$$\pi(x) = 2\partial_-\phi(x), \quad \partial_+\phi(x) = \frac{1}{4}\mu^2\partial_-^{-1}\phi(x). \tag{4}$$

∂_-^{-1} = inverse derivative. The quantum solution of the field equation (3)

$$\phi(x) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+) e^{-\frac{i}{2}k^+ x^- - \frac{i}{2}\frac{\mu^2}{k^+}x^+} + a^\dagger(k^+) e^{\frac{i}{2}k^+ x^- + \frac{i}{2}\frac{\mu^2}{k^+}x^+}\right], \tag{5}$$

with the Fock (creation and annihilation) operators satisfying

$$\left[a(k^+), a^\dagger(l^+)\right] = \delta(k^+ - l^+), \;\; a(k^+)|0\rangle = 0. \tag{6}$$

Equivalently, the field commutation relation at equal LF time $(z^- = x^- - y^-)$

$$\left[\phi(x^+, x^-), \phi(x^+, y^-)\right] = \int_0^{\infty} \frac{dk^+}{4\pi k^+} \left[e^{-\frac{i}{2}k^+(z^- - i\epsilon^-)} - e^{\frac{i}{2}k^+(z^- + i\epsilon^-)}\right] \equiv -\frac{i}{4}\epsilon(z^-), \tag{7}$$

$\epsilon(z^-)$ being the sign function. The LF Hamiltonian and the momentum operator are

$$P^- = \frac{1}{2} \int_{-\infty}^{+\infty} dx^- T^{+-}(x) = \int_0^{+\infty} dk^+ \frac{\mu^2}{k^+} a^\dagger(k^+) a(k^+),$$

$$P^+ = \frac{1}{2} \int_{-\infty}^{+\infty} dx^- T^{++}(x) = \int_0^{+\infty} dk^+ k^+ a^\dagger(k^+) a(k^+), \tag{8}$$

$$T^{++} = 4 : \partial_- \phi \partial_- \phi : \quad , \quad T^{+-} = \mu^2 : \phi^2 : . \tag{9}$$

From (5) we directly find

$$\theta(x) \equiv 2\partial_+\phi(x) = -i \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \frac{\mu^2}{k^+} \big[a(k^+)e^{-i\hat{k}\cdot x} - a^\dagger(k^+)e^{i\hat{k}\cdot x}\big],$$

$$\pi(x) = 2\partial_-\phi(x) = -i \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} k^+ \big[a(k^+)e^{i\hat{k}\cdot x} - a^\dagger(k^+)e^{i\hat{k}\cdot x}\big]. \qquad (10)$$

Various two-point correlation functions from the three field operators above:

$$D_0^{(+)}(z) = \langle 0|\phi(x)\phi(y)|0\rangle, \qquad (11)$$

$$D_1^{(+)}(z) = \langle 0|\phi(x)\pi(y)|0\rangle, \qquad (12)$$

$$D_2^{(+)}(z) = \langle 0|\phi(x)\theta(y)|0\rangle, \qquad (13)$$

$$D_i^{(+)}(z) = i \int_0^\infty \frac{dk^+}{4\pi} f_i(k^+) e^{-\frac{i}{2}k^+(z^- - i\epsilon^-) - \frac{i}{2}\frac{\mu^2}{k^+}(z^+ - i\epsilon^+)},$$

$$f_0(k^+) = -\frac{i}{k^+},\ f_1(k^+) = 1,\ f_2(k^+) = \frac{\mu^2}{k^{+2}}. \tag{14}$$

The small imaginary parts in time and space coordinates (implicitly present also in (5) and 10)) introduced in order that the integrals exist. The resultant exponential damping factors replace the role of test functions.

The integrals explicitly evaluated in terms of the (modified) Bessel functions $J_\nu(z), N_\nu(z), K_\nu(z), \nu = 0, 1$. As in the conventional SL theory, the first one logarithmically diverges for vanishing mass μ. The second is given by

$$D_1^{(+)}(z) = -\theta(z^2)\frac{\mu}{4}\sqrt{\frac{z^+}{z^-}}i\Big[J_1\big(\mu\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(\mu\sqrt{z^2}\big)\Big] + \\ - \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{\mu}{4\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(\mu\sqrt{-z^2}\big). \quad (15)$$

$D_2^{(+)}$ obtained from $D_1^{(+)}$ by the interchange $x^+ \leftrightarrow x^-$. The important observation: both $D_1^{(+)}$ and $D_2^{(+)}$ have a non-vanishing massless limit:

$$D_1^{(+)}(x-y;\mu^2=0) = \frac{1}{2\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \\ D_2^{(+)}(x-y;\mu^2=0) = \frac{1}{2\pi}\frac{1}{(x^+ - y^+ - i\epsilon^+)}. \quad (16)$$

Technically, this is due to the behaviour of the Bessel function $K_1(z) \sim \frac{1}{z}$ for small value of z (the same is true for $N_1(z)$ in the timelike region), so that

$$\frac{\mu}{2\pi}\sqrt{-\frac{z^{\mp}}{z^{\pm}}}K_1\big(\mu\sqrt{-z^+z^-}\big) \tag{17}$$

has the finite massless limit (Bergknoff 1977) $1/(2\pi z^{\pm})$.

The results (16) suggest that there must exist massless analogs of the fields $\phi(x), \pi(x), \theta(x)$, whose correlation functions reproduce the above massless limits of the massive correlators.

From the LF massless Klein-Gordon equation

$$\partial_+\partial_-\tilde{\phi}(x) = 0 : \tag{18}$$

one should expect a general solution of the form

$$\tilde{\phi}(x) = \tilde{\phi}(x^+) + \tilde{\phi}(x^-). \tag{19}$$

The mass dependence resides only in the plane-wave factor[1], the massless limit of the massive solution (5) yields $\tilde{\phi}(x^-)$:

$$\tilde{\phi}(x^-) = \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \left[a(k^+)e^{-\frac{i}{2}k^+x^-} + a^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right]. \tag{20}$$

But where is the second piece $\tilde{\phi}(x^+)$? Also contained in (5)! To show this, change the variables as (more correctly at the classical level) $k^+ = \frac{\mu^2}{k^-}$. One

[1]The measure in the LF momentum integrals is, contrary to the space-like form of the theory, mass-independent (Leutwyler, Klauder and Streit 1972).

obtains

$$\phi(x) = \int_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} \Big[\frac{\mu}{k^-} a\big(\frac{\mu^2}{k^-}\big) e^{-\frac{i}{2}\frac{\mu^2}{k^-}x^- - \frac{i}{2}k^- x^+} + \frac{\mu}{k^-} a^\dagger\big(\frac{\mu^2}{k^-}\big) e^{\frac{i}{2}\frac{\mu^2}{k^-}x^- + \frac{i}{2}k^- x^+} \Big]. \tag{21}$$

The Fock commutators in terms of the new variables:

$$\Big[\frac{\mu}{k^-} a\big(\frac{\mu^2}{k^-}\big), \frac{\mu}{l^-} a^\dagger\big(\frac{\mu^2}{l^-}\big) \Big] = \frac{\mu^2}{k^- l^-} \delta\big(\frac{\mu^2}{k^-} - \frac{\mu^2}{l^-}\big) = \delta(k^- - l^-). \tag{22}$$

Since the rhs does not depend on mass and hence survives the massless limit, this should be true for the lhs as well. Thus (classical level first)

$$\lim_{\mu \to 0} \frac{\mu}{k^-} a\big(\frac{\mu^2}{k^-}\big) \equiv \tilde{a}(k^-) \tag{23}$$

is non-vanishing, with the properties

$$\left[\tilde{a}(k^-), \tilde{a}^\dagger(l^-)\right] = \delta(k^- - l^-), \quad \left[a(k^+), \tilde{a}^\dagger(l^-)\right] = 0. \tag{24}$$

The massless limit in (21) and in $\pi(x), \theta(x)$ (10):

$$\tilde{\phi}(x^+) = \int\limits_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} \left[\tilde{a}(k^-)e^{-\frac{i}{2}k^-x^+} + \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right], \tag{25}$$

$$\tilde{\theta}(x) = -i\int\limits_0^{+\infty} \frac{dk^-}{\sqrt{4\pi k^-}} k^-\left[\tilde{a}(k^-)e^{-\frac{i}{2}k^-x^+} - \tilde{a}^\dagger(k^-)e^{\frac{i}{2}k^-x^+}\right], \tag{26}$$

$$\tilde{\pi}(x) = -i\int\limits_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} k^+\left[a(k^+)e^{-\frac{i}{2}k^+x^-} - a^\dagger(k^+)e^{\frac{i}{2}k^+x^-}\right]. \tag{27}$$

The basic commutators, following from (6) and (24):

$$[\tilde{\phi}(x^-), \tilde{\phi}(y^-)] = -\frac{i}{4}\epsilon(x^- - y^-), \quad [\tilde{\phi}(x^+), \tilde{\phi}(y^+)] = -\frac{i}{4}\epsilon(x^+ - y^+). \quad (28)$$

Thus, the second half of the solution of the wave equation has been recovered from the massive solution.

The variables k^+ and k^- coincide - analogous to the SL case $k^0 = |k^1|$. In the LF case $k^- = k^+$ directly (both are positive-definite).

Consistency check: the two-point functions calculated from the massless fields should coincide with the massless limit of the massive functions. Indeed the case for $D_1^{(+)}(z)$ and $D_2^{(+)}(z)$:

$$\tilde{D}_1^{(+)}(z) = \langle 0|\tilde{\phi}(x^-)\tilde{\pi}(y^-)|0\rangle = \frac{1}{2\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \tag{29}$$

since the $\tilde{\phi}(x^+)$ term does not contribute (see Eq.(24)). The massive $D_0^{(+)}(z)$ can be evaluated for small mass μ, it diverges as lnμ.

Note: $D_0^{(+)}(z)$ calculated from the massless solution is ill defined (infinite) since μ has been already set to zero. Upon introducing the infrared cutoffs $\lambda^+ = \lambda^- \equiv \lambda$ in the corresponding integrals,

$$\tilde{D}_0^{(+)}(z) = \int_\lambda^\infty \frac{dk^-}{4\pi k^-} e^{-\frac{i}{2}k^-(z^+ - i\epsilon^+)} + \int_\lambda^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(z^- - i\epsilon^-)}, \tag{30}$$

the same (regularized) $\ln\lambda$ divergent behaviour found (see below).

The massless limit of the LF momentum and Hamiltonian:

One expects that these operators survive the massless limit since for a consistent quantum theory we have to find the Heisenberg equations

$$2i\partial_+\phi(x) = -[P^-, \phi(x)], \;\; 2i\partial_-\phi(x) = -[P^+, \phi(x)]. \quad (31)$$

The massless limit of the momentum operator (9) straightforward, while the change of variables (21) necessary for the Hamiltonian. Using the Fock commutators (24), the resultant operators

$$P^+ = \int_0^{+\infty} dk^+ k^+ a^\dagger(k^+) a(k^+), \quad P^- = \int_0^{+\infty} dk^+ k^- \tilde{a}^\dagger(k^-) \tilde{a}(k^-) \quad (32)$$

are easily seen to generate the Heisenberg equations

$$2i\partial_{+}\phi(x^{+}) = -[P^{-},\phi(x^{+})], \quad 2i\partial_{-}\phi(x^{-}) = -[P^{+},\phi(x^{-})], \tag{33}$$

since $[P^{+},\phi(x^{+})] = [P^{-},\phi(x^{-})] = 0$. On the other hand, the massless limit of the LF Hamiltonian density $T^{+-}(x)$ itself vanishes. This is in agreement with the underlying conformal symmetry of the massless theory, which requires (see the next section)

$$\mathrm{Tr}\mathrm{T}^{\mu\nu} = \mathrm{T}^{\mu}{}_{\mu} = \mathrm{T}^{+-} = 0. \tag{34}$$

II.2 Massless light front fermion field.

The situation with fermions simpler since no infrared divergencies present. The *massive* field equation (the 2D version of the Dirac equation):

$$i\gamma^\mu \partial_\mu \psi(x) = m\psi(x). \tag{35}$$

In the LF variables decomposes into a dynamical and a constraint equation:

$$2i\partial_+ \psi_2(x) = m\psi_1(x), \quad 2i\partial_- \psi_1(x) = m\psi_2(x) \Rightarrow \psi_1(x) = \frac{m}{2i}\partial_-^{-1}\psi_2(x). \tag{36}$$

Chiral representation for the Dirac matrices: $\gamma^\pm = \gamma^0 \pm \gamma^1, \gamma^0 = \sigma^1, \gamma^1 = i\sigma^2, \gamma^5 = \gamma^0\gamma^1$, where σ^1, σ^2 are Pauli matrices. For $m = 0$:

$$\psi_2(x) = \psi_2(x^-), \quad \psi_1(x) = \psi_1(x^+). \tag{37}$$

So ψ_1 is a zero mode (x^-–independent quantity) that seemingly needs to be quantized independently on the surface $x^- = 0$ (McCartor 1992,1995). This assumption however does not generate a consistent theoretical framework (two evolution parameters, negative momenta of new modes).

INSTEAD: Start again from the massive fields in the momentum representation and study its massless limit

The equations (36) solved by

$$\psi_2(x) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} \mathrm{d}p^+ \left[b(p^+) e^{-\frac{i}{2}p^+ x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} + d^\dagger(p^+) e^{\frac{i}{2}p^+ x^- + \frac{i}{2}\frac{m^2}{p^+}x^+} \right], \quad (38)$$

$$\psi_1(x) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} \mathrm{d}p^+ \frac{m}{p^+} \left[b(p^+) e^{-\frac{i}{2}p^+ x^- - \frac{i}{2}\frac{m^2}{p^+}x^+} - d^\dagger(p^+) e^{\frac{i}{2}p^+ x^- + \frac{i}{2}\frac{m^2}{p^+}x^+} \right], (39)$$

with the Fock operators obeying

$$\{b(p^+), b^\dagger(q^+)\} = \{d(p^+), d^\dagger(q^+)\} = \delta(p^+ - q^+). \tag{40}$$

NOTE: the massless limit of ψ_2 well defined

The two-point functions $S^{(+)}_{\alpha\beta}(x-y) = \langle 0|\psi_\alpha(x)\psi^\dagger_\beta(y)|0\rangle, \alpha, \beta = 1, 2$, are expressed in terms of the scalar-field functions as

$$S^{(+)}_{22}(z) = -iD^{(+)}_1(z), \;\; S^{(+)}_{11}(z) = -iD^{(+)}_2(z), \;\; S^{(+)}_{12}(z) = mD^{(+)}_0(z) \tag{41}$$

with $\mu \to m$.

Due to the behaviour of the $K_1(z)$ and $N_1(z)$ functions for small z analogous to (17), the massless limit of the functions $S^{(+)}_{22}$ and $S^{(+)}_{11}$ is finite:

$$S_{22}^{(+)}(x-y;m=0)=\frac{1}{2i\pi}\frac{1}{(x^- - y^- - i\epsilon^-)}, \tag{42}$$

$$S_{11}^{(+)}(x-y;m=0)=\frac{1}{2i\pi}\frac{1}{(x^+ - y^+ - i\epsilon^+)}. \tag{43}$$

The mixed two-point function $S_{12}^{(+)}$ vanishes for $m=0$.

The result (42) easily obtained directly from the massless $\psi_2(x^-)$ (well defined simply by setting $m=0$ in the plane wave factors of Eq.(38)).

ALSO: Eq.(43) clearly indicates that there should exist a massless fermion-field component $\psi_1(x^+)$ whose 2-point function is given by (43). Change of variables $p^- = \frac{m^2}{p^+}$ again required first to be able to perform the limit.

The anticommutators in terms of the new variables:

$$\left\{\frac{m}{p^-}b\Big(\frac{m^2}{p^-}\Big), \frac{m}{q^-}b^\dagger\Big(\frac{m^2}{q^-}\Big)\right\} = \frac{m^2}{p^-q^-}\delta\Big(\frac{m^2}{p^-} - \frac{m^2}{q^-}\Big) = \delta(p^- - q^-). \qquad (44)$$

By the same reasoning as for the scalar field, we conclude that

$$\lim_{m\to 0}\frac{m}{p^-}b\Big(\frac{m^2}{p^-}\Big) \equiv \tilde{b}(p^-), \quad \lim_{m\to 0}\frac{m}{p^-}d\Big(\frac{m^2}{p^-}\Big) \equiv \tilde{d}(p^-) \qquad (45)$$

are non-vanishing. Hence

$$\tilde{\psi}_2(x^-) = \frac{1}{\sqrt{4\pi}}\int_0^{+\infty} dp^+\Big[b(p^+)e^{-\frac{i}{2}p^+x^-} + d^\dagger(p^+)e^{\frac{i}{2}p^+x^-}\Big], \qquad (46)$$

$$\tilde{\psi}_1(x^+) = \frac{1}{\sqrt{4\pi}} \int_0^{+\infty} dp^- \left[\tilde{b}(p^-) e^{-\frac{i}{2}p^- x^+} - \tilde{d}^\dagger(p^-) e^{\frac{i}{2}p^- x^+}\right], \tag{47}$$

$$\{\tilde{b}(p^-), \tilde{b}^\dagger(q^-)\} = \{\tilde{d}(p^-), \tilde{d}^\dagger(q^-)\} = \delta(p^- - q^-), \tag{48}$$

$$\{\tilde{b}(p^-), b^\dagger(q^+)\} = \{\tilde{d}(p^-), d^\dagger(q^+)\} = 0. \tag{49}$$

The expected form of the field anticommutators obtained:

$$\left\{\tilde{\psi}_1(x^+), \tilde{\psi}_1{}^\dagger(y^+)\right\} = \delta(x^+ - y^+), \quad \left\{\tilde{\psi}_2(x^-), \tilde{\psi}_2{}^\dagger(y^-)\right\} = \delta(x^- - y^-). \tag{50}$$

The two-point function calculated from the massless $\tilde{\psi}_1(x^+)$ coincides with the massless limit (43) of the massive 2-point function.

- A SIMPLE AND CONSISTENT FRAMEWORK ESTABLISHED

No new variables have to be intoduced, the necessary information contained in the original massive solution.

The massless vector current $j^\mu = \overline{\psi}\gamma^\mu\psi$ from the massless fields (46,47):

$$j^+(x^-) = \lim_{\epsilon^- \to 0} \left[\tilde{\psi_2}^\dagger (x^- + \frac{\epsilon^-}{2}) \tilde{\psi}_2 (x^- - \frac{\epsilon^-}{2}) + H.c. \right] = 2 : \tilde{\psi}_2^\dagger (x^-) \tilde{\psi}_2 (x^-) : \quad (51)$$

$$j^-(x^+) = \lim_{\epsilon^+ \to 0} \left[\tilde{\psi_1}^\dagger (x^+ + \frac{\epsilon^+}{2}) \tilde{\psi}_1 (x^+ - \frac{\epsilon^+}{2}) + H.c. \right] = 2 : \tilde{\psi}_1^\dagger (x^+) \tilde{\psi}_1 (x^+) : \quad (52)$$

Solvable models are based on free Heisenberg fields $\Rightarrow$ the above derivation of the two-dimensional massless LF fermion fields opens the avenue for the genuine light-front solution of this class of models

II.3 LF bosonization.

A remarkable property of two-dimensional field theory: fermion fields can be represented in terms of boson variables (Coleman, Mandelstam,..).

Our quantization of the massless LF scalar and fermion fields: formulate the bosonization property in a genuine LF form. Since the massless $\phi(x)$ and $\psi(x)$ fields decompose as

$$\phi(x) = \phi(x^+) + \phi(x^-),\ \psi^T(x) = (\psi_1(x^+), \psi_2(x^-))$$

the demonstration of bosonization very simple (tilde omitted henceforth)

Start with $\psi_2(x^-)$. Assume that it can be represented as

$$\varphi_2(x^-) = C : e^{i\alpha\phi(x^-)} : \;\; = C e^{i\alpha\phi^{(-)}(x^-)} e^{i\alpha\phi^{(+)}(x^-)}. \tag{53}$$

Adjust the constants C and α in such a way, that two φ_2 with different arguments anticommute and $\varphi_2(x^-)$, $\varphi_2^\dagger(y^-)$ satisfy the anticommutation relation (50). The first condition fixes α to the value $\alpha = 2\sqrt{\pi}$:

Form the product $\varphi_2(x^-)\varphi_2(y^-)$ and perform the necessary commutations to obtain the opposite order of the operators. This, using the operator identity $\exp A \exp B = \exp\left(-\frac{1}{2}[A, B]\right) \exp\left(A + B\right)$, generates the expression

$$\varphi_2(x^-)\varphi_2(y^-) = e^{-\alpha^2\left(D_0^{(+)}(x^- - y^-) - D_0^{(+)}(y^- - x^-)\right)} \varphi_2(y^-)\varphi_2(x^-). \tag{54}$$

The two commutator functions $D_0^{(+)}(\pm(x^- - y^-))$, where

$$D_0^{(+)}(x^- - y^-) = \left[\phi^{(+)}(x^-), \phi^{(-)}(y^-)\right] = \int\limits_0^\infty \frac{dk^+}{4\pi k^+} e^{-\frac{i}{2}k^+(x^- - y^- - i\epsilon^-)}, \quad (55)$$

individually diverge, but upon introducing the infrared cutoff λ

$$D_0^{(+)}(z^-) = -\frac{1}{4\pi} \ln\left[\frac{\lambda}{2} e^{\gamma_E}(iz^- + \epsilon^-)\right], \quad z^- = x^- - y^-, \quad (56)$$

the divergent parts cancel in (54) producing the sign function $\epsilon(x^- - y^-)$. With $\alpha = 2\sqrt{\pi}$, the net result is $e^{i\pi\epsilon(x^- - y^-)} = -1$ for all x^-, y^-.
This is the required anticommutativity.

To prove the the second property, form the anticommutator

$$A(x^-,y^-) \equiv \varphi_2(x^-)\varphi_2^\dagger(y^-) + \varphi_2^\dagger(y^-)\varphi_2(x^-) = C^2 \times$$
$$\times\left[e^{4\pi D_0^{(+)}(x^- - y^-)} : \varphi_2(x^-)\varphi_2^\dagger(y^-) : + e^{4\pi D_0^{(+)}(y^- - x^-)} : \varphi_2^\dagger(y^-)\varphi_2(x^-) :\right]. \tag{57}$$

Taking into account the explicit form of the infrared-regularized $D_0^{(+)}$ function and the fact that two normal-ordered expressions in (57) actually coincide, we find

$$A(x^-,y^-) = \frac{2}{i\lambda e^{\gamma_E}}\left[\frac{1}{z^- - i\epsilon^-} - \frac{1}{z^- + i\epsilon^-}\right] : \varphi_2(x^-)\varphi_2^\dagger(y^-) := \frac{4\pi}{\lambda e^{\gamma_E}}\delta(x^- - y^-). \tag{58}$$

Used: the term in the square bracket is equal to $2i\pi\delta(z^-)$. The operator part on the rhs has reduced to unity due to the presence of this delta-function. It follows that the rescaled operator

$$\hat{\varphi}_2(x^-) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^-)} e^{i2\sqrt{\pi}\phi^{(+)}(x^-)} \tag{59}$$

obeys the correct anticommutation relation (50) and represents the bosonized form of the fermion field $\psi_2(x^-)$. The construction of the second component $\varphi_1(x^+)$ is completely paralel, with $x^- \to x^+$, etc. Thus

$$\hat{\varphi}_1(x^+) = \sqrt{\frac{\lambda e^{\gamma_E}}{4\pi}} e^{i2\sqrt{\pi}\phi^{(-)}(x^+)} e^{i2\sqrt{\pi}\phi^{(+)}(x^+)}. \tag{60}$$

The vector current in the bosonic form: Inserting the bosonic form (59) into (51), one gets a product of four exponential operators, which is NOT in the normal order. Commute the two middle terms to normal-order the expression, one generates a term $e^{\alpha^2 D_0^{(+)}(\epsilon^-)}$, which according to (56) behaves as $1/\epsilon^-$. This singularity is canceled by the terms from the exponential, linear in ϵ^-. No vacuum subtractions are needed, since the second (conjugate) term in (51) cancels them automatically. The net result

$$j^+(x^-) = \frac{2}{\sqrt{\pi}}\partial_-\phi(x^-), \;\; j^-(x^+) = \frac{2}{\sqrt{\pi}}\partial_+\phi(x^+). \tag{61}$$

The second current component was obtained in a completely analogous way. The boson representation correctly reproduces **the Schwinger term** in both current-current commutators:

$$\left[j^+(x^-), j^+(y^-)\right] = \frac{i}{\pi}\partial_x\delta(x^- - y^-), \;\; \left[j^-(x^+), j^-(y^+)\right] = \frac{i}{\pi}\partial_x\delta(x^+ - y^+). \tag{62}$$

Similarly, for **the scalar densities** (no singularities - no splitting)

$$\overline{\psi}(x)\psi(x) = \psi_1^\dagger(x^+)\psi_2(x^-) + \psi_2^\dagger(x^-)\psi_1(x^+), \tag{63}$$

$$\overline{\psi}(x)\gamma^5\psi(x) = \psi_1^\dagger(x^+)\psi_2(x^-) - \psi_2^\dagger(x^-)\psi_1(x^+). \tag{64}$$

one obtains ($\phi(x) = \phi(x^+) + \phi(x^-)$)

$$\overline{\psi}(x)\psi(x) = \frac{\lambda e^{\gamma_E}}{4\pi}\cos\left(2\sqrt{\pi}\phi(x)\right), \;\; \overline{\psi}(x)\gamma^5\psi(x) = i\frac{\lambda e^{\gamma_E}}{4\pi}\sin\left(2\sqrt{\pi}\phi(x)\right). \tag{65}$$

Conclusion: the LF version of bosonization yields the results known from the SL theory in a simple and transparent form.

useful for bosonized Thirring model or sine-Gordon model?

conformal field theory in 2D

III. SL AND LF DERIVATIVE COUPLING MODEL

The simplest model – illustration of the derivation of the correct Hamiltonians and SL-LF comparison

The classical Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overset{\leftrightarrow}{\partial_{\mu}} \Psi - m\overline{\Psi}\Psi + \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_{\mu}\phi J^{\mu}, \quad J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi. \quad (66)$$

For $\mu = 0$ known as the Schroer's model, for axial vector current interaction as Rothe-Stamatescu model ($m = 0, \mu \neq 0$).

Euler-Lagrange eqs.

$$\begin{aligned} i\gamma^{\mu}\partial_{\mu}\Psi &= m\Psi + g\partial_{\mu}\phi\gamma^{\mu}\Psi, \\ \partial_{\mu}\partial^{\mu}\phi + \mu^2\phi &= g\partial_{\mu}J^{\mu}. \end{aligned} \quad (67)$$

Convention: capital Greek letters - interacting Heisenberg fields, small - free fields

Classically, the vector current is conserved, $\partial_\mu J^\mu(x) = 0 \Rightarrow$ free scalar field (not guaranteed at the quantum level)

classical solution of the Dirac eq.

$$\Psi(x) = e^{ig\phi(x)}\psi(x), \quad i\gamma^\mu\partial_\mu\psi(x) = m\psi(x). \tag{68}$$

irrespectively if scalar field is free or interacting

$\phi(x)$ quantized by $\left[a(k^1), a^\dagger(l^1)\right] = \delta(k^1 - l^1)$,
using notation $\hat{p}.x \equiv \omega(p^1)t - p^1x^1, \omega(p^1) = \sqrt{p_1^2 + \mu^2}$

$$\phi(x) \quad = \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{dk^1}{\sqrt{2\omega(k^1)}}\left[a(k^1)e^{-i\hat{k}.x} + a^\dagger(k^1)e^{i\hat{k}.x}\right]$$

$$\equiv \quad \phi^{(+)}(x) + \phi^{(-)}(x). \tag{69}$$

The free massive fermion field quantized as

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp^1 \left[u(p^1)b(p^1)e^{-i\hat{p}.x} + v(p^1)d^\dagger(p^1)e^{i\hat{p}.x}\right],$$
$$u^\dagger(p^1) = (\sqrt{p^+}, \sqrt{p^-}), \quad v(p^1)^\dagger = (\sqrt{p^+}, -\sqrt{p^-}),$$
$$p^\pm = E(p^1) \pm p^1, E(p^1) = \sqrt{p_1^2 + m^2}$$
$$\{b(p^1), b^\dagger(q^1\} = \{d(p^1), d^\dagger(q^1)\} = \delta(p^1 - q^1). \tag{70}$$

Remark Approach here a bit heuristic, operators have to be regularized, finite-volume treatment.

At the quantum level, the solution has to be regularized:

$$\Psi(x) = Z^{1/2}(\epsilon) e^{-ig\phi^{(-)}(x)} e^{-ig\phi^{(+)}(x)} \psi(x), \tag{71}$$

where $Z(\epsilon) = \exp\left\{g^2\left[\phi^{(+)}(x - \frac{\epsilon}{2}), \phi^{(-)}(x + \frac{\epsilon}{2})\right]\right\} = \exp\left\{-ig^2 D_0^{(+)}(\epsilon)\right\}$.

Apply the point-splitting regularization to the interacting currents:

$$\begin{aligned} J^\mu(x) &= s \lim_{\epsilon \to 0} \frac{1}{2}\Big\{ Z(\epsilon) \overline{\psi}(x + \frac{\epsilon}{2}) e^{ig\phi^{(-)}(x+\frac{\epsilon}{2})} e^{ig\phi^{(+)}(x+\frac{\epsilon}{2})} \gamma^\mu \\ &\times e^{-ig\phi^{(-)}(x-\frac{\epsilon}{2})} e^{-ig\phi^{(+)}(x-\frac{\epsilon}{2})} \psi(x - \frac{\epsilon}{2}) + H.c. \Big\} = \\ &=: \overline{\psi}(x) \gamma^\mu \psi(x) : + \frac{g}{2\pi} \partial^\mu \phi(x). \end{aligned} \tag{72}$$

Symmetric limit ($s\lim_{\epsilon\to 0}\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2} = \frac{1}{2}g^{\mu\nu}$), free field relation

$$\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\psi(x-\frac{\epsilon}{2}) =: \overline{\psi}(x)\gamma^\mu\psi(x) : -\frac{i}{\pi}\frac{\epsilon^\mu}{\epsilon^2} \tag{73}$$

No need to subtract the VEV part by hand if one defines the (free) current as a hermitian sum

$$j^\mu(x) = \lim_{\epsilon\to 0}\frac{1}{2}\Big[\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\psi(x-\frac{\epsilon}{2}) + \overline{\psi}(x-\frac{\epsilon}{2})\gamma^\mu\psi(x+\frac{\epsilon}{2})\Big] \tag{74}$$

$Z(\epsilon)$ cancelled by the opposite factor from commuting two middle terms

The quantum vector current received a correction (”anomaly”, $\partial_\mu j^\mu = 0$),

$$\partial_\mu J^\mu(x) = \frac{g}{2\pi}\partial_\mu\partial^\mu\phi(x) \tag{75}$$

The only effect: finite mass ”renormalization”:

$$\partial_\mu \partial^\mu \phi(x) + \tilde{\mu}^2 \phi(x) = 0, \quad \tilde{\mu}^2 = \frac{\mu^2}{1 - \frac{g^2}{2\pi}}. \tag{76}$$

Axial-vector current is conserved (if $m = 0$):

$$J_5^\mu(x) =: \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x) : -\frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x). \tag{77}$$

Conjugate momenta directly

$$\Pi_\phi = \partial_0\phi(x) - gJ^0, \quad \Pi_\Psi = \frac{i}{2}\Psi^\dagger(x), \quad \Pi_{\Psi^\dagger} = -\frac{i}{2}\Psi(x). \tag{78}$$

The Hamiltonian

$$H = H_{0B} + H^{'},$$

$$H_{0B} = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[\frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2\Big],$$

$$H^{'} = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \big[-i\Psi^\dagger\alpha^1\partial_1\Psi + m\Psi^\dagger\gamma^0\Psi + g\partial_1\phi J^1\big]. \quad (79)$$

In the kinetic term the free field $\psi(x)$ taken,

$$H_{0F} = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \big[-i\psi^\dagger\alpha^1\partial_1\psi + m\psi^\dagger\gamma^0\psi\big],$$

$$H_{0F} = \int_{-\infty}^{+\infty} \mathrm{d}p^1 E(p^1)\Big[b^\dagger(p^1)b(p^1) + d^\dagger(p^1)d(p^1)\Big],$$

$$H_{0B} = \int_{-\infty}^{+\infty} \mathrm{d}p^1 \omega(p^1) a^\dagger(p^1) a(p^1), \quad \omega(p^1) = \sqrt{p_1^2 + \mu^2}, \tag{80}$$

The interacting Hamiltonian becomes

$$H_{int} = \frac{g}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k^1 \left[c^\dagger(k^1) a(k^1) + a^\dagger(k^1) c(k^1) + a^\dagger(k^1) c^\dagger(k^1) + a(k^1) c(k^1) \right]. \tag{81}$$

where the composite boson operators satisfying $[c(k^1), c^\dagger(l^1)] = \delta(k^1 - l^1)$ correspond to the vector current

$$j^\mu(x) = -\frac{i}{\sqrt{2}\pi} \int \frac{dk^1}{\sqrt{2k^0}} k^\mu \left\{ c(k^1) e^{-i\hat{k}.x} - c^\dagger(k^1) e^{i\hat{k}.x} \right\}, \tag{82}$$

The Hamiltonian non-diagonal, a Bogoliubov transformation necessary for $m = 0$ implemented by means of a unitary operator $U = \exp(iS)$ with

$$S(\gamma) = -i \int_{-\infty}^{+\infty} \mathrm{d}k^1 \gamma(k) \left[c^\dagger(k^1) a^\dagger(-k^1) - c(k^1) a(-k^1) \right]. \tag{83}$$

The physical vacuum found as

$$|\Omega\rangle = N \exp\left[\int_{-\infty}^{+\infty} \mathrm{d}k^1 \gamma(g) c^\dagger(-k^1) a^\dagger(k^1) \right] |0\rangle. \tag{84}$$

nontrivial vacuum structure
Also: momentum operator contains interaction!

THE LF TREATMENT

Covariant Lagrangian in terms of LF space-time and field variables

$$\mathcal{L}_{lf} = i{\Psi_2}^\dagger \overleftrightarrow{\partial}_+ \Psi_2 + i{\Psi_1}^\dagger \overleftrightarrow{\partial}_- \Psi_1 - m({\Psi_1}^\dagger\Psi_2 + {\Psi_2}^\dagger\Psi_1) + \\ + 2\partial_+\phi\partial_-\Phi - \frac{1}{2}\mu^2\phi^2 - g\partial_+\phi J^+ - g\partial_-\phi J^-, \quad (85)$$

Euler-Lagrange equations in the component form read

$$2i\partial_+\Psi_2 = m\Psi_1 + 2g\partial_+\phi\Psi_2, \qquad 2i\partial_-\Psi_1 = m\Psi_2 + 2g\partial_-\phi\Psi_1 \quad (86)$$

Inserting the constraint into the Lagrangian leads to the free LF Lagrangian and Hamiltonian!

$$P^- = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2}\Big[m\big({\psi_1}^\dagger\psi_2 + {\psi_2}^\dagger\psi_1\big) + \mu^2\phi^2\Big] \quad (87)$$

CLEAR CONTRADICTION BETWEEN THE SL and LF FORMALISMS!

WAY OUT:

The solution of the field equations not taken into account!

The solution tells us that there is no "independent" interacting field – it is composed out of free fields. The free fields are the true physical degrees of freedom and the Lagrangian has to be re-expressed in terms of them first (analogously to inserting a constraint into Lagrangian), then calculate conjugate momenta and derive the Hamiltonian.

NOTE: this is not the same as inserting the field equation (Dirac eq., Klein-Gordon eq.) into $\mathcal{L}$ – the latter leads to vanishing Lagrangian (extremum of the action)

Dirac eq. implies knowledge of $\gamma^\mu \partial_\mu \Psi$, knowing the solution implies knowing $\partial_\mu \Psi$

Inserting the solution of the Dirac eq. of the DCM in the form

$$\partial_\mu \Psi(x) = -ig\partial_\mu \phi(x)\Psi(x) + e^{-ig\phi(x)}\partial_\mu \psi(x) \tag{88}$$

into $\mathcal{L}$ yields

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \psi - m\overline{\psi}\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2, \tag{89}$$

i.e. the interaction part got cancelled! Free form of the Lagrangian, free fields and conjugate momenta ($\Pi_\psi = i\psi^\dagger, \Pi_\phi = \partial_0\phi$) and the Hamiltonian

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \big[-i\psi^\dagger\alpha^1\partial_1\psi + m\psi^\dagger\gamma^0\psi + \frac{1}{2}\Pi_\phi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2\big], \tag{90}$$

which is just the sum of free Hamiltonians of the massive scalar and fermion fields. Correct Heisenberg equations generated with this Hamiltonian:

$$-i\partial_0\Psi(x) = [H, \Psi(x)] \tag{91}$$

Physical vacuum coincides with the Fock vacuum. The only trace of the interacting theory is the non-canonical form of the anticommutation relation of the interacting fermion field and the form of the correlation functions. The latter expressed in terms of the correlation functions of free fields,

$$\langle vac|\Psi_\alpha(x)\overline{\Psi}_\beta(y)|vac\rangle = \langle 0| : e^{-ig\phi(x)} : \psi_\alpha(x)\overline{\psi}_\beta(y) : e^{ig\phi(y)} : |0\rangle =$$
$$= e^{g^2 D_0^{(+)}(x-y)} S_{\alpha\beta}^{(+)}(x-y), \tag{92}$$

where

$$D_0^{(+)}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle,$$

$$D_0^{(+)}(z) = -\frac{1}{4}\theta(z^2)\Big[N_0(\mu\sqrt{z^2}) + \\ + i\,\mathrm{sgn}(z^0)J_0(\mu\sqrt{z^2})\Big] + \frac{1}{2\pi}\theta(-z^2)K_0(\mu\sqrt{-z^2}). \quad (93)$$

The fermionic two-point function is

$$S_{\alpha\beta}^{(+)}(x-y) = \langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle,$$
$$S_{\alpha\beta}^{(+)}(z) = \Big(i\gamma^\mu\partial_\mu + m\Big)_{\alpha\beta} D^{(+)}(z). \quad (94)$$

Explictly,

$$S^{(\pm)}(z) = \frac{i}{2\pi}(i\gamma^\mu\partial_\mu^x + m)\int d^2p\,\delta(p^2-m^2)\theta(\pm p^0)e^{\pm ip.z} =$$

$$= \frac{i}{4\pi} \int \frac{dp}{E(p)} \begin{pmatrix} m & p^- \\ p^+ & m \end{pmatrix} e^{\pm i\hat{p}.z} \tag{95}$$

Remark: B. Schroer used this model in 1961 (Fort. Physik 1) to illustrate the concept of "infraparticle". His results are ok if one considers his interacting Lagrangian – which however is not the true Lagrangian of the model

The LF analysis proceeds analogously:

$$\mathcal{L}_{lf} = i{\Psi_2}^\dagger \overset{\leftrightarrow}{\partial}_+ \Psi_2 + i{\Psi_1}^\dagger \overset{\leftrightarrow}{\partial}_- \Psi_1 - m({\Psi_1}^\dagger\Psi_2 + {\Psi_2}^\dagger\Psi_1) +$$
$$+ 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_+\phi J^+ - g\partial_-\phi J^-, \tag{96}$$

Field equations

$$2i\partial_+\Psi_2 = m\Psi_1 + 2g\partial_+\phi\Psi_2,$$
$$2i\partial_-\Psi_1 = m\Psi_2 + 2g\partial_-\phi\Psi_1 \tag{97}$$

solved by

$$\Psi_2(x) = e^{-ig\phi(x)}\psi_2(x), \quad 2i\partial_+\psi_2 = m\psi_1$$
$$\Psi_1(x) = e^{-ig\phi(x)}\psi_1(x), \quad 2i\partial_-\psi_1 = m\psi_2. \tag{98}$$

Inserting these solutions into the LF Lagrangian yields the free one:

$$\mathcal{L}_{lf} = i{\psi_2}^\dagger \overleftrightarrow{\partial_+} \psi_2 + i{\psi_1}^\dagger \overleftrightarrow{\partial_-} \psi_1 - m({\psi_1}^\dagger\psi_2 + {\psi_2}^\dagger\psi_1) + 2\partial_+\phi\partial_-\phi - \frac{1}{2}\mu^2\phi^2. \tag{99}$$

Free Hamiltonian follows:

$$P^- = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{2} \Big[m\big({\psi_1}^\dagger \psi_2 + {\psi_2}^\dagger \psi_1\big) + \mu^2 \phi^2 \Big]. \tag{100}$$

The same as before. Reason: no kinetic term in LF Hamiltonian present by construction

Correlation functions coincide with those from the space-like treatment.

$$\langle 0|\Psi(x)\overline{\Psi}(y)|0\rangle = e^{-\frac{g^2}{\pi} D_0^{(+)}(x-y)} S^{(+)}(x-y). \tag{101}$$

$$S_{22}(x-y) = \langle 0|\psi_2(x)\psi_2^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon)},$$

$$S_{11}(x-y) = \langle 0|\psi_1(x)\psi_1^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi}\frac{m^2}{p^{+2}} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon)},$$

$$S_{12}(x-y) = \langle 0|\psi_1(x)\psi_2^\dagger(y)|0\rangle = \int_0^\infty \frac{dp^+}{4\pi}\frac{m}{p^+} e^{-\frac{i}{2}p^+(x^- - y^- - i\epsilon) - \frac{i}{2}\frac{m^2}{p^+}(x^+ - y^+ - i\epsilon)}. \quad (102)$$

Note that we have introduced the small imaginary parts in time and space coordinates. This step dictated by the mathematical consistency. Without the damping factors the integrals would not exist as mathematical objects [Gradshteyn and Ryzhik]. The scalar-field function is

$$D_0^{(+)}(z) = mS_{12}^{(+)}(z). \tag{103}$$

The fermion-field functions are

$$\begin{aligned} S_{22}^{(+)}(z) = & - & \theta\big(z^2\big)\frac{m}{4}\sqrt{\frac{z^+}{z^-}}\Big[J_1\big(m\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] + \\ & + & \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{2\pi}\sqrt{-\frac{z^+}{z^-}}K_1\big(m\sqrt{-z^2}\big), \\ S_{11}^{(+)}(z) = & & \theta\big(z^2\big)\frac{m}{4}\sqrt{\frac{z^-}{z^+}}\Big[J_1\big(m\sqrt{z^2}\big) - i\,\mathrm{sgn}(z^+)N_1\big(m\sqrt{z^2}\big)\Big] - \end{aligned}$$

$$
\begin{aligned}
& - \quad \theta\big(-z^2\big)\mathrm{sgn}(z^+)\frac{im}{2\pi}\sqrt{-\frac{z^-}{z^+}}K_1\big(m\sqrt{-z^2}\big), \\
S_{12}^{(+)}(z) = & - \quad \theta\big(z^2\big)\frac{m}{4}\Big[N_0\big(m\sqrt{z^2}\big) + i\,\mathrm{sgn}(z^+)J_0(m\sqrt{z^2})\Big] + \\
& + \quad \theta\big(-z^2\big)\frac{m}{2\pi}K_0\big(m\sqrt{-z^2}\big) \qquad (104)
\end{aligned}
$$

Small imaginary parts in the arguments with appropriate sign are understood. Calculation of the analogous correlation functions in the conventional theory is more complicated and requires a clever change of variables [Bogoliubov and Shirkov].

The scalar-field correlation function diverges for $\mu = 0$ in both schemes. LF calculation with the massless fermion field inconsistent (yields vanishing $S_{11}^{(+)}(z)$.) The $m = 0$ limit of the LF fermion correlation function coincides with the SL case.

MASSIVE ROTHE-STAMATESCU MODEL – A FEW REMARKS

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m\overline{\Psi}\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_\mu\phi J_5^\mu, \quad J_5^\mu = \overline{\Psi}\gamma^\mu\gamma^5\Psi. \tag{105}$$

Non-trivial physics found in literature (Belvedere and Rodrigues in a series of papers) : axial anomaly, anomalous dimension of the fermion field, relation to the massive Thirring and sine-Gordon models...

Based on the definition of the vector current:

$$j_\epsilon^\mu(x) = \overline{\Psi}(x+\epsilon)\gamma^\mu\Psi(x)\exp\left(ig\int_x^{x+\epsilon} dy_\lambda \epsilon^{\lambda\nu}\partial_\nu\phi(y)\right) - VEV. \tag{106}$$

”an extended treatment”

”conservative approach”:

Field equations:

$$i\gamma^\mu \partial_\mu \Psi = m\Psi + g\partial_\mu \phi \gamma^\mu \gamma^5 \Psi,$$
$$\partial_\mu \partial^\mu \phi + \mu^2 \phi^2 = g\partial_\mu J_5^\mu = 2img\overline{\Psi}\gamma^5\Psi. \quad (107)$$

Scalar field is no longer free, Dirac eq. seems to have an operator solution similar to the one from the DCM:

$$\Psi(x) = e^{-ig\gamma^5\phi(x)}\psi(x). \quad (108)$$

Check:

$$i\gamma^\mu \partial_\mu \Psi(x) = i\gamma^\mu \big[-ig\gamma^5 \partial_\mu \phi(x) \Psi(x) + e^{-ig\gamma^5\phi(x)} \partial_\mu \psi(x) \big] =$$
$$= g\partial_\mu \phi(x) \gamma^\mu \gamma^5 \Psi(x) + e^{+ig\gamma^5\phi(x)} i\gamma^\mu \partial_\mu \psi(x), \quad (109)$$

where $i\gamma^\mu\partial_\mu\psi = m\psi$. The sign in the last exponential is opposite due to $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$.

Thus, the massive RS model is not exactly solvable. The original massless RS model (Rothe and Stamatescu, Annals of Physics 1977): The massless axial current is conserved, hence scalar field is free. Dirac eq. is exactly solvable but inserting the solution to the Lagrangian generates the free Hamiltonian. Similar to the massive derivative-coupling model.

iterative (perturbative) approach in the Heisenberg picture?

IV. MASSLESS THIRRING MODEL

W.E. Thirring, Annals of Physics 3 (1958) 91, 631 citations

Thirring model played important role in history of QFT (see Wightman's Cargese lectures and Klaiber's paper)

operator solution due to B. Klaiber (Boulder 1967), n-point correlation functions constructed, basis of the Coleman's (perturbative) bosonization

all aspects clarified?

not quite true: a series of papers by Faber and Ivanov (discovery of a broken phase claimed based on Nambu – Jona-Lasinio BCS-like Ansatz for the ground state)

similar conclusions done by Fujita et al. using the Bethe Ansatz solution systematic Hamiltonian study based on the model's solvability not given so far (however some ideas and methods by S. Korenblit are close to ours – LM and P. Grange, PLB (2013))

Classical Lagrangian density

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^{\mu} \overset{\leftrightarrow}{\partial_{\mu}} \Psi - \frac{1}{2}gJ_{\mu}J^{\mu}, \quad J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi. \tag{110}$$

Field equations and current conservation

$$\begin{aligned} i\gamma^{\mu}\partial_{\mu}\Psi(x) &= gJ^{\mu}(x)\gamma_{\mu}\Psi(x), \\ \partial_{\mu}J^{\mu}(x) &= 0. \end{aligned} \tag{111}$$

The general solution is

$$\Psi(x) = e^{-i(g/\sqrt{\pi})\left(\alpha j(x) - \beta\gamma^{5}\tilde{j}(x)\right)}\psi(x),$$

$$\gamma^\mu \partial_\mu \psi(x) = 0 \tag{112}$$

with $\alpha + \beta = 1$. "Potentials" $j(x)$ and $\tilde{j}(x)$ connected to the free vector current by $\partial_\mu j(x) = -\sqrt{\pi} j_\mu(x),\ \partial_\mu \tilde{j}(x) = \sqrt{\pi}\epsilon_{\mu\nu} j^\nu(x)$. This corresponds to replacing $J^\mu(x)$ by $j^\mu(x)$ in the field equation – rather restrictive, does not represent the most general quantum solution. The latter can be obtained as follows. Consider the $\beta = 0$ case for simplicity

$$\Psi(x) = e^{i(g/\sqrt{\pi})J(x)}\psi(x) \tag{113}$$

with the unknown potential $J(x)$ of the interacting current $J^\mu(x)$, i.e. defining $\partial_\mu J(x) = -\sqrt{\pi} J_\mu(x)$. Regularizing (113) like in the DCM model and calculating the corresponding current using the point-split product of the above $\Psi^\dagger$ and Ψ, we find

$$J^\mu(x) =: \overline{\psi}(x)\gamma^\mu\psi(x) : + \frac{g}{2\pi} J^\mu(x) \Rightarrow$$

$$J^\mu(x) = G(g) j^\mu(x),\ G(g) = (1 - \frac{g}{2\pi})^{-1}. \tag{114}$$

Interacting current = rescaled free current. Potential consequences for Coleman's bosonization.

Fourier representation

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int dp^1 \{ b(p^1) u(p^1) e^{-ip.x} + d^\dagger(p^1) v(p^1) e^{ip.x} \},\ \ p^0 = |p^1|$$
$$\{ b(p^1), b^\dagger(q^1) \} = \{ d(p^1), d^\dagger(q^1) \} = \delta(p^1 - q^1),$$
$$b(k^1)|0\rangle = d(k^1)|0\rangle = 0. \tag{115}$$

The spinors $u(p^1), v(p^1)$ are $m = 0$ limits of the massive spinors,

$$u^\dagger(p^1) = \big(\theta(-p^1), \theta(p^1)\big),\ \ v^\dagger(p^1) = \big(-\theta(-p^1), \theta(p^1)\big). \tag{116}$$

Vector current $j^\mu = (:\psi^\dagger\psi:, :\psi^\dagger\alpha^1\psi:)$:

$$j^0(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}p^1\int_{-\infty}^{+\infty}\mathrm{d}q^1\Big\{f_0(p^1,q^1)\Big[b^\dagger(p^1)b(q^1) - d^\dagger(p^1)d(q^1)\Big]e^{i(\hat{p}-\hat{q}).x}\Big] +$$

$$+\, g_0(p^1,q^1)\Big[b^\dagger(p^1)d^\dagger(q^1)e^{i(\hat{p}+\hat{q}).x} + d(p^1)b(q^1)e^{-i(\hat{p}+\hat{q}).x}\Big]\Big\},$$

$$j^1(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}p^1\int_{-\infty}^{+\infty}\mathrm{d}q^1\Big\{g_0(p^1,q^1)\Big[b^\dagger(p^1)b(q^1) - d^\dagger(p^1)d(q^1)\Big]e^{i(\hat{p}-\hat{q}).x}\Big] +$$

$$+\, f_0(p^1,q^1)\Big[b^\dagger(p^1)d^\dagger(q^1)e^{i(\hat{p}+\hat{q}).x} + d(p^1)b(q^1)e^{-i(\hat{p}+\hat{q}).x}\Big]\Big\},$$

$$f_0(p^1,q^1) = \theta(p^1)\theta(q^1) + \theta(-p^1)\theta(-q^1),$$

$$g_0(p^1,q^1) = \theta(p^1)\theta(q^1) - \theta(-p^1)\theta(-q^1). \quad (117)$$

can be represented in terms of composite fermion operators

$$j^{\mu}(x) = -\frac{i}{\sqrt{2}\pi}\int \frac{dk^1}{\sqrt{2k^0}} k^{\mu}\{c(k^1)e^{-i\hat{k}.x} - c^{\dagger}(k^1)e^{i\hat{k}.x}\}, \tag{118}$$

where (Fourier transform)

$$c(k^1) = \frac{i}{\sqrt{k^0}}\int dp^1\{\theta\big(p^1k^1\big)\big[b^{\dagger}(p^1)b(p^1+k^1) - d^{\dagger}(p^1)d(p^1+k^1)\big] + \\ +\epsilon(p^1)\theta\big(p^1(p^1-k^1)\big)d(k^1-p^1)b(p^1)\big\}. \tag{119}$$

Canonical Fock commutation relation follow

$$\big[c(p^1), c^{\dagger}(q^1)\big] = \delta(p^1-q^1), \;\; c(k^1)|0\rangle = 0. \tag{120}$$

Problem: infrared divergence – the two-point correlation function of a massless scalar field in D=1+1 is divergent,

$$D_0^{(+)}(x-y;\mu=0) = \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{4\pi}\int \frac{dk^1}{|k^1|} e^{-i\hat{k}.x}. \tag{121}$$

True ground state of the massless Thirring model:

Hamiltonian in the usual treatment (kinetic term taken as built from free field) is

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[-i\psi^\dagger \alpha^1 \partial_1 \psi + \frac{1}{2} g \big(j^0 j^0 - j^1 j^1 \big) \Big] \tag{122}$$

Not correct. Insert the operator solution to the Lagrangian first:

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu \Big[-\frac{ig}{\sqrt{\pi}} \partial_\mu j \Psi + e^{-\frac{ig}{\sqrt{\pi}} j} \partial_\mu \psi \Big] - \frac{g}{2} j_\mu j^\mu. \tag{123}$$

The first term in the bracket combines with the interaction term reversing its sign. The correct Hamiltonian is

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[- i\psi^\dagger \alpha^1 \partial_1 \psi - \frac{1}{2} g \big(J^0 J^0 - J^1 J^1 \big) \Big]. \tag{124}$$

Fock representation: the free Hamiltonian is

$$H_0 = \int_{-\infty}^{+\infty} \mathrm{d}p^1 |p^1| \Big[b^\dagger(p^1) b(p^1) + d^\dagger(p^1) d(p^1) \Big]. \tag{125}$$

The interacting Hamiltonian greatly simplifies in terms of composite operators $c(k^1), c^\dagger(k^1)$:

$$H_g = G^2(g)\frac{g}{\pi}\int_{-\infty}^{+\infty} dk^1|k^1|\Big[c^\dagger(k^1)c^\dagger(-k^1) + c(k^1)c(-k^1)\Big]. \qquad (126)$$

Obviously $H = H_0 + H_g$ is not diagonal and $|0\rangle$ is not its eigenstate.

DETAILS:

H_0 satisfies

$$\big[H_0, c(k^1)\big] = -|k^1|c(k^1), \quad \big[H_0, c^\dagger(k^1)\big] = |k^1|c^\dagger(k^1). \qquad (127)$$

Remark: mathematically correct treatment requires cut-offs or test functions to have well defined quantities, here the approach a little heuristic (but checked in a finite volume)

To diagonalize H, define the operator T with the same commutation property:

$$T = \int_{-\infty}^{+\infty} \mathrm{d}k^1 |k^1| c^\dagger(k^1) c(k^1),$$

$$\left[T, c(k^1)\right] = -|k^1| c(k^1), \quad \left[T, c^\dagger(k^1)\right] = |k^1| c^\dagger(k^1). \tag{128}$$

Consider now the unitary operator U,

$$U = e^{iS}, \quad S = -\frac{i}{2} \int_{-\infty}^{+\infty} \mathrm{d}p^1 \gamma(p^1) \left[c^\dagger(p^1) c^\dagger(-p^1) - c(p^1) c(-p^1)\right]. \tag{129}$$

Form new free and interacting Hamiltonians

$$\hat{H}_0 = H_0 - T, \quad \hat{H}_g = H_g + T. \tag{130}$$

By construction, due to $\left[S, \hat{H}_0\right] = 0$, $\hat{H}_0$ is invariant with respect to U:

$$\hat{H}_0 \to e^{iS}\hat{H}_0 e^{-iS} = \hat{H}_0 + i\left[S, \hat{H}_0\right] + \ldots = \hat{H}_0. \tag{131}$$

On the other hand, $\hat{H}_{int}$ transforms non-trivially due to

$$\left[S, c(k^1)\right] = i\gamma(k^1)c^\dagger(-k^1), \quad \left[S, c^\dagger(k^1)\right] = i\gamma(k^1)c(-k^1), \;\; \gamma(-k^1) = \gamma(k^1). \tag{132}$$

Using the operator identity $e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + $ $+\frac{1}{3!}[A, [A, [A, B]]] + \ldots$:

$$e^{iS}c(k^1)e^{-iS} = c(k^1) + i\big(i\gamma(k^1)\big)c^\dagger(-k^1) + \frac{i^2}{2}\big(i\gamma(k^1)\big)^2 c(k^1) + \frac{i^3}{3!}\big(i\gamma(k^1)\big)^3 c^\dagger(-k^1$$

$$+\ldots. \quad (1$$

Thus

$$c(k^1) \to e^{iS} c(k^1) e^{-iS} = c(k) \cosh\gamma(k^1) - c^\dagger(-k^1) \sinh\gamma(k^1),$$
$$c^\dagger(k^1) \to e^{iS} c^\dagger(k^1) e^{-iS} = c^\dagger(k^1) \cosh\gamma(k^1) - c(-k^1) \sinh\gamma(k^1) \quad (1.134)$$

It follows

$$\hat{H}_g \to e^{iS} \hat{H}_g e^{-iS} =$$
$$\int\limits_{-\infty}^{+\infty} \mathrm{d}k^1 |k^1| \Big\{ \Big[c^\dagger(k^1) c^\dagger(-k^1) + c(k^1) c(-k^1) \Big] \Big[G^2 \frac{g}{2\pi} \Big(\cosh^2\gamma(k^1) + \sinh^2\gamma(k^1) \Big) -$$
$$- \cosh\gamma(k^1) \sinh\gamma(k^1) \Big] -$$

$$-c^\dagger(k^1)c(k)\Big[4G^2\frac{g}{2\pi}\sinh\gamma(k^1)\cosh\gamma(k^1) - \Big(\cosh^2\gamma(k^1) + \sinh^2\gamma(k^1)\Big)\Big] -$$
$$-\delta(0)\Big[2\sinh\gamma(k^1)\cosh\gamma(k^1) + \sinh^2\gamma(k^1)\Big]\Big\}. \tag{135}$$

The last (divergent) term removed by normal ordering. Diagonal form if $\gamma(k^1) = \gamma_d = \frac{1}{2}\mathrm{arctanh}(2G(g)\frac{g}{\pi})$.

Thus we have achieved

$$e^{iS}\big(\hat{H}_0 + \hat{H}_g\big)e^{-iS}|0\rangle = 0 \tag{136}$$

and $|\Omega \rightarrow= e^{-iS}|0\rangle$ is the new vacuum state. Explicitly,

$$|\Omega\rangle = \exp\Big[-\frac{1}{2}\gamma_d\int\limits_{-\infty}^{+\infty}\mathrm{d}p^1\big[c^\dagger(p^1)c^\dagger(-p^1) - c(p^1)c(-p^1)\big]\Big]|0\rangle. \tag{137}$$

A coherent state of pairs of effective bosons (bilinear in fermion Fock operators) with zero total momentum:

$$P^1|\Omega\rangle = 0, \quad P^1 = \int_{-\infty}^{+\infty} \mathrm{d}p^1 p^1 \left[b^\dagger(p^1)b(p^1) + d^\dagger(p^1)d(p^1)\right]. \tag{138}$$

The vacuum $|\Omega\rangle$ is invariant under $U(1)$ and $U_A(1)$ transformations (i.e. carries vanishing charge and axial charge):

$$U(\alpha)|\Omega\rangle = |\Omega\rangle, \quad U(\alpha) = e^{i\alpha Q}, \quad Q = \int_{-\infty}^{+\infty} \mathrm{d}q^1 \left[b^\dagger(q^1)b(q^1) - d^\dagger(q^1)d(q^1)\right],$$

$$V(\beta)|\Omega\rangle = |\Omega\rangle, \;\; V(\beta) = e^{i\beta Q_5}, \;\; Q_5 = \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \epsilon(q^1)\big[b^\dagger(q^1)b(q^1) - d^\dagger(q^1)d(q^1)\big].$$

The vacuum state $|\Omega\rangle$ corresponds to the symmetric phase (is invariant with respect to axial-vector transformations, i.e. no chiral symmetry breaking) – in contradiction with the results of Faber and Ivanov true vacuum should be an eigenstate of the full Hamiltonian! $|\Omega\rangle$ is such a state

TWO-POINT FUNCTION

Correlation functions calculated from the known operator solution

$$\Psi(x) = e^{(-ig/\sqrt{\pi})j^+(x)}\psi(x)e^{(-ig/\sqrt{\pi})j^-(x)}. \tag{139}$$

$\psi(x)$ is the free massless fermion field and $j^\pm(x)$ are the positive and negative-frequency parts of the integrated current $j(x) = j^{(+)}(x) + j^{(-)}(x)$:

$$j^{(+)}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \frac{c^\dagger(q^1)}{\sqrt{2|q^1|}} \left[e^{i\hat{q}.x} - \theta\big(\lambda - |q^1|\big) \right],$$

$$j^{(-)}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \mathrm{d}q^1 \frac{c(q^1)}{\sqrt{2|q^1|}} \left[e^{-i\hat{q}.x} - \theta\big(\lambda - |q^1|\big) \right]. \tag{140}$$

The infared regularization necessary to have meaningful objects. The scale λ introduced. The two-point function defined as

$$C_2(x-y) = \langle vac|\Psi(x)\overline{\Psi}(y)|vac\rangle. \tag{141}$$

What is $|vac\rangle$? As a rule, the perturbative vacuum state taken. Commuting the fermion operators through the exponentials and the exponentials themselves, one arrives at

$$C_2(x-y) = e^{\frac{g^2}{\pi} D_0^{(+)}(x-y)} e^{-2g\left[D_0^{(+)}(y-x)+\gamma^5 \tilde{D}_0^{(+)}(y-x)\right]} \langle 0|\psi(x)\overline{\psi}(y)|0\rangle. \quad (142)$$

Here, with $\mu = e^{\gamma_E}\lambda$,

$$D^{(+)}(x) = \frac{1}{2\pi}\int \frac{dk}{2|k|}\theta(|k|-\lambda)e^{-ik.x} = -\frac{1}{4\pi}\ln\big(-\mu^2x^2+ix^0\epsilon\big) \quad (143)$$

Calculation with $|\Omega\rangle$ more complicated:

$$\langle\Omega|\Psi(x)\overline{\Psi}(y)|\Omega\rangle = F_2(x-y;\kappa)C_2(x-y). \quad (144)$$

The function $F_2(x-y;\kappa)\to 1$ for $\kappa\to 0$.

The other aspects:

canonical quantization may not always be valid for interacting fields:

$$\left\{\Psi(x), \Psi^\dagger(y)\right\} = Z^{-1}\delta(x-y), \quad Z^{-1} = \exp\left(g^2 D^{(+)}(0)\right)$$

Next: LF version of the massless Thirring model, operator solution of the Thirring-Wess model