Zero mode issue in the minus-minus component calculation of the transition form factors in the light-front dynamics

Bailing Ma

Argonne National Lab

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LFQCD seminar

Outline

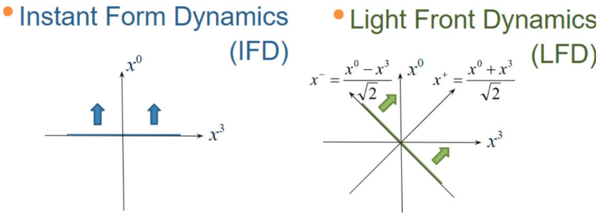
1. Meson-photon transition form factors
   - covariant results
   - light-front dynamics results
2. Minus-minus component of the meson-photon transition amplitude
   - pure LFD calculation
   - intermediate interpolation method calculation
   - full interpolation calculation
3. Spurious form factors in the light-front dynamics

Meson-photon transition form factors

Meson-photon transition form factors (TFFs)

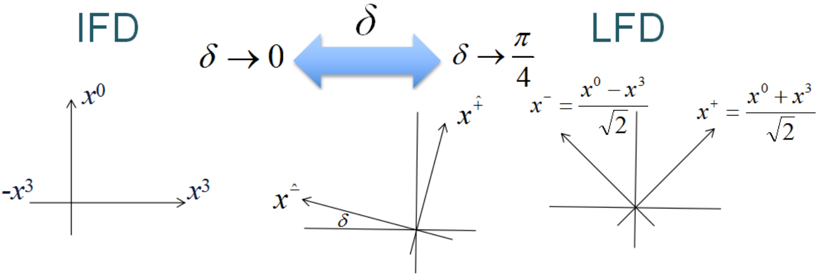
- They are the simplest exclusive processes involving the strong interaction.
- Can be factorized into the hard scattering amplitude and the meson distribution amplitude (DA). The meson DAs provide essential information about the non-perturbative dynamics of a QCD bound state.
- Time-like TFFs beyond normal threshold ($q^2 > 4m^2$) can provide very important information for understanding quark confinement.

Light-front dynamics (LFD)



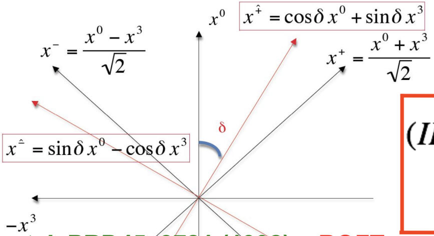
- LFD has additional symmetry compared to IFD such as boost invariance: the individual x^+-ordered amplitudes are frame independent while the x^0-ordered amplitudes are not.
- In LFD, vacuum fluctuations are suppressed except for zero modes.
- However, LFD is strictly in Minkowski space.
- So things that are usually developed in the IFD based on a Euclidean formulation such as temperature dependent QFTs, and lattice QCD, are not generally accessible in LFD.

Interpolating between the IFD and LFD



$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Interpolating between the IFD and LFD



$$(IFD) \quad 0 \leq \delta \leq \frac{\pi}{4} \quad (LFD)$$
$$1 \geq \quad C \equiv \cos(2\delta) \quad \geq 0$$

K. Hornbostel, PRD45, 3781 (1992) – RQFT
C.Ji and S.Rey, PRD53,5815(1996) – Chiral Anomaly
C.Ji and C. Mitchell, PRD64,085013 (2001) – Poincare Algebra
C.Ji and A. Suzuki, PRD87,065015 (2013) – Scattering Amps
C.Ji, Z. Li and A. Suzuki, PRD91, 065020 (2015) – EM Gauges
Z.Li, M. An and C.Ji, PRD92, 105014 (2015) – Spinors
C.Ji, Z.Li, B.Ma and A.Suzuki, PRD98, 036017(2018) – QED
B.Ma and C.Ji, PRD104, 036004(2021) – QCD_{1+1}

Simple scalar model in 1+1 space-time dimensions

- Gives a clear example of spurious form factors which may arise in GPDs as well.
- We can directly compute the time-like form factors without resorting to analytic continuation, both by a manifestly covariant calculation and in LFD by using a $q^+ \neq 0$ frame.
- Paves the road for future analysis using 't Hooft model, i.e. QCD_2, which has confinement due to the Coulomb interaction becoming linear in 1+1 space-time dimension.

Meson $\rightarrow \gamma^*\gamma^*$ transition form factor in 1+1-d scalar model: manifestly covariant calculation

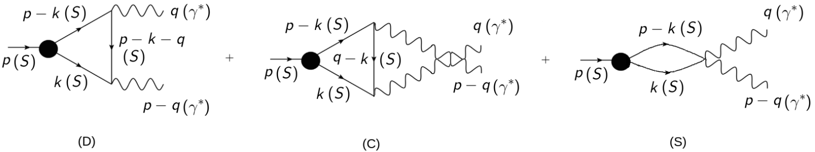


Figure: One-loop covariant Feynman diagrams that contribute to the $S \rightarrow \gamma^*\gamma^*$ transition form factor

The total amplitude consists of these three Feynman diagrams, i.e., the direct (D), crossed (C), and the seagull (S) diagrams, where p is the momentum of the incident meson, while q is the momentum of the emitted photon. As a result of momentum conservation, $q' = p - q$ is the momentum of the final photon.

From gauge invariance argument, we can know that the total amplitude $\Gamma^{\mu\nu}$ is of the form

$$\Gamma^{\mu\nu} = F(q^2, q'^2)\left(g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}\right), \tag{1}$$

which satisfies both

$$q_{\mu}\left(g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}\right) = 0 \tag{2}$$

and

$$q'_{\nu}\left(g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}\right) = 0, \tag{3}$$

so that the form factor can be obtained by

$$F(q^2, q'^2) = \frac{\Gamma^{\mu\nu}}{g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}}. \tag{4}$$

The amplitude $\Gamma^{\mu\nu}$ is calculated as such, following the Feynman rules for the scalar field theory.

$$\begin{aligned}\Gamma^{\mu\nu} =&\Gamma^{\mu\nu}_D + \Gamma^{\mu\nu}_C + \Gamma^{\mu\nu}_S \\ =&ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{(2p-2k-q)^\mu (p-2k-q)^\nu}{((p-k-q)^2-m^2)((p-k)^2-m^2)(k^2-m^2)} \right. \\ &+ \frac{(q-2k)^\mu (p-2k+q)^\nu}{((p-k)^2-m^2)(k^2-m^2)((q-k)^2-m^2)} \\ &\left. + \frac{-2g^{\mu\nu}}{((p-k)^2-m^2)(k^2-m^2)} \right\}, \end{aligned} \tag{5}$$

where the coupling constant of the simple scalar model g_s is fixed from the normalization condition. For simplicity, we take all the intermediate scalar particles' mass to be m and their charge to be e, but it can be easily generalized to unequal mass/charge cases. The initial scalar meson has mass M.

We finally obtain

$$F_{cov}(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \int_0^1 dx \int_0^{1-x} dy (1-2y) \left(\frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \right), \quad (6)$$

where

$$\Delta_1 = x(x-1)q^2 + 2x(x+y-1)q\cdot q' + (x+y)(x+y-1)q'^2 + m^2, \quad (7)$$

$$\Delta_2 = x(x-1)q'^2 + 2x(x+y-1)q\cdot q' + (x+y)(x+y-1)q^2 + m^2. \quad (8)$$

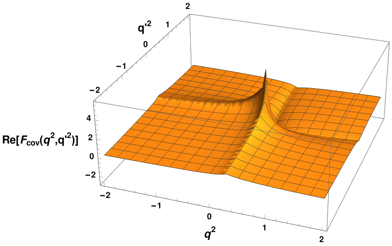
Doing the x and y integrations, we get the analytic formula for the transition form factor,

$$F(q^2, q'^2) = \frac{e^2 g_s}{4\pi} \times \frac{(2-\omega-\gamma'-\gamma)\frac{\sqrt{\omega}}{\sqrt{1-\omega}}\tan^{-1}\left(\frac{\sqrt{\omega}}{\sqrt{1-\omega}}\right) + (\gamma-\gamma'-\omega)\frac{\sqrt{1-\gamma'}}{\sqrt{\gamma'}}\tan^{-1}\left(\frac{\sqrt{\gamma'}}{\sqrt{1-\gamma'}}\right) + (\gamma'-\gamma-\omega)\frac{\sqrt{1-\gamma}}{\sqrt{\gamma}}\tan^{-1}\left(\frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}\right)}{m^4\left[4\omega\gamma'\gamma + \omega^2 + (\gamma'-\gamma)^2 - 2\omega(\gamma'+\gamma)\right]}, \quad (9)$$

where $\gamma = \frac{q^2}{4m^2}$, $\gamma' = \frac{q'^2}{4m^2}$, and $\omega = \frac{M^2}{4m^2}$.

Numerical results for the TFF covering spacelike and timelike regions

Now, taking $m = 0.25\ GeV$, $M = 0.14\ GeV$, and normalizing the form factor so that $F(q^2 = 0, q'^2 = 0) = 1$ (thus fixing g_s), we show below the numerical results of the form factor as a function of q^2 and q'^2.



Meson $\to \gamma^*\gamma^*$ transition form factor in 1+1-d scalar model: LFD calculation

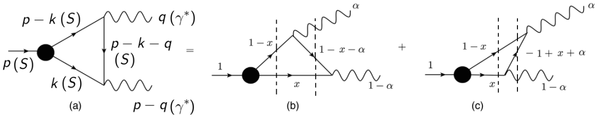


Figure: (Take the direct diagram as an example). The covariant diagram (a) is sum of the two LF x^+-ordered diagrams (b): Valence (V), $0 < x < 1 - \alpha < 1$ and (c): Non-valence (NV), $0 < 1 - \alpha < x < 1$.

- α is the momentum fraction q^+/p^+, and x is the momentum fraction k^+/p^+.
- Usually people assume each individual LFTO diagram is only of the gauge invariant form, i.e. $\Gamma_i^{\mu\nu} = f_i(q^2, q'^2)\,(g^{\mu\nu} q \cdot q' - q'^\mu q^\nu)$, and obtain the LFTO contributions to the form factor from the ++ current only: $f_{(b)} = \frac{\Gamma^{++}_{(b)}}{g^{++} q \cdot q' - q'^+ q^+}$, $f_{(c)} = \frac{\Gamma^{++}_{(c)}}{g^{++} q \cdot q' - q'^+ q^+}$.

Democracy of the LF components

However, we can define the contributions using other components

$$f_i^{(++)} = \frac{\Gamma_i^{++}}{g^{++} q \cdot q' - q'^{+} q^{+}}, \tag{10}$$

$$f_i^{(+-)} = \frac{\Gamma_i^{+-}}{g^{+-} q \cdot q' - q'^{+} q^{-}}, \tag{11}$$

$$f_i^{(-+)} = \frac{\Gamma_i^{-+}}{g^{-+} q \cdot q' - q'^{-} q^{+}}, \tag{12}$$

$$f_i^{(--)} = \frac{\Gamma_i^{--}}{g^{--} q \cdot q' - q'^{-} q^{-}}. \tag{13}$$

where i represents $D(b)$, $D(c)$, $C(b)$, $C(c)$, or S.

Minus-minus component of the meson-photon transition amplitude

Meson $\to \gamma^*\gamma^*$ TFF in 1+1-d scalar model: "$--$" component of LFTO calculation

The minus minus component of the transition amplitude is

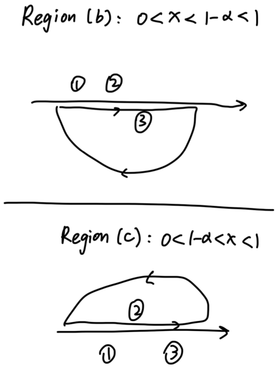
$$\begin{aligned}\Gamma^{--} &= \Gamma_D^{--} + \Gamma_C^{--} \\ &= ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(2p-2k-q)^-(p-2k-q)^-}{((p-k-q)^2-m^2)((p-k)^2-m^2)(k^2-m^2)} \\ &+ ie^2 g_s \int \frac{d^2k}{(2\pi)^2} \frac{(q-2k)^-(p-2k+q)^-}{((p-k)^2-m^2)(k^2-m^2)((q-k)^2-m^2)}. \end{aligned} \tag{14}$$

There is no seagull term contribution for the $--$ current. Now plugging in the kinematics, we get (I'm only showing the direct diagram calculation and omitting the repetitive crossed diagram calculation)

$$\begin{aligned}\Gamma_D^{--} = \frac{ie^2 g_s}{4\pi^2} \int dk^+ \int dk^- \left(\frac{M^2}{2p^+} - 2k^- + \frac{q'^2}{2(1-\alpha)p^+}\right)\left(-2k^- + \frac{q'^2}{2(1-\alpha)p^+}\right) \\ \cdot \left(2(p-k-q)^+(p-k-q)^- - m^2\right)^{-1}\left(2(p-k)^+(p-k)^- - m^2\right)^{-1} \\ \cdot \left(2k^+k^- - m^2\right)^{-1}. \end{aligned} \tag{15}$$

Now let us do the integration $\int dk^-$.
There are 3 poles

$$
\begin{aligned}
k_1^- &= p^- - q^- - \frac{m^2 - i\varepsilon}{2(p-k-q)^+} \\
k_2^- &= p^- - \frac{m^2 - i\varepsilon}{2(p-k)^+} \\
k_3^- &= \frac{m^2 - i\varepsilon}{2k^+}
\end{aligned}
\tag{16}
$$



where the k_2^- pole is located at the upper half plane, k_3^- pole is at lower half plane, while k_1^- pole depending on the sign of $1-x-\alpha$, when $1-x-\alpha>0$, it is at upper plane, and we call this region (b). For region (b), we enclose the contour for lower half plane and catch pole 3. When $1-x-\alpha<0$, it is at lower plane, and we call this region (c). For region (c), we enclose the contour for upper half plane and catch pole 2.

Calculating the pole residue, we get

$$
\begin{aligned}
&\Gamma^{--}_{D(b)} \\
&= \frac{ie^2 g_s}{4\pi^2}(-2\pi i)\int dk^+ \left(\frac{M^2}{2p^+} + \frac{q'^2}{2(1-\alpha)p^+} - 2k_3^-\right)\left(\frac{q'^2}{2(1-\alpha)p^+} - 2k_3^-\right) \\
&\cdot \left[2\,(p-k-q)^+\,2\,(p-k)^+\,2k^+\,(k_3^- - k_1^-)\,(k_3^- - k_2^-)\right]^{-1} \\
&= \frac{e^2 g_s}{2\pi}p^+ \int_0^{1-\alpha} dx \left(\frac{M^2}{2p^+} + \frac{q'^2}{2(1-\alpha)p^+} - \frac{m^2}{xp^+}\right)\left(\frac{q'^2}{2(1-\alpha)p^+} - \frac{m^2}{xp^+}\right) \\
&\cdot \Big[2p^+\,(1-x-\alpha)\,2p^+\,(1-x)\,2p^+ x \\
&\cdot \frac{1}{2p^+}\left(\frac{m^2}{x} - \frac{q'^2}{1-\alpha} + \frac{m^2}{1-x-\alpha}\right)\frac{1}{2p^+}\left(\frac{m^2}{x} - M^2 + \frac{m^2}{1-x}\right)\Big]^{-1} \\
&= \frac{e^2 g_s}{4\pi p^+ p^+}\int_0^{1-\alpha} dx \left(\frac{M^2}{2} + \frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x}\right)\left(\frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x}\right) \\
&\cdot \left[(1-x-\alpha)(1-x)x\left(\frac{m^2}{x} + \frac{m^2}{1-x-\alpha} - \frac{q'^2}{1-\alpha}\right)\left(\frac{m^2}{x} + \frac{m^2}{1-x} - M^2\right)\right]^{-1}
\end{aligned}
\qquad (17)
$$

and

$$
\begin{aligned}
&\Gamma^{--}_{D(c)} \\
&= \frac{ie^2 g_s}{4\pi^2}(2\pi i)\int dk^+ \left(\frac{M^2}{2p^+} + \frac{q'^2}{2(1-\alpha)p^+} - 2k_2^-\right)\left(\frac{q'^2}{2(1-\alpha)p^+} - 2k_2^-\right) \\
&\cdot \left[2\left(p-k-q\right)^+ 2\left(p-k\right)^+ 2k^+ \left(k_2^- - k_1^-\right)\left(k_2^- - k_3^-\right)\right]^{-1} \\
&= \frac{-e^2 g_s}{2\pi} p^+ \int_{1-\alpha}^{1} dx \left(\frac{M^2}{2p^+} + \frac{q'^2}{2(1-\alpha)p^+} - \frac{M^2}{p^+} + \frac{m^2}{(1-x)p^+}\right) \\
&\cdot \left(\frac{q'^2}{2(1-\alpha)p^+} - \frac{M^2}{p^+} + \frac{m^2}{(1-x)p^+}\right) \\
&\cdot \Big[2p^+\left(1-x-\alpha\right) 2p^+\left(1-x\right) 2p^+ x \\
&\cdot \frac{1}{2p^+}\left(-\frac{m^2}{1-x} + M^2 - \frac{q'^2}{1-\alpha} + \frac{m^2}{1-x-\alpha}\right)\frac{1}{2p^+}\left(M^2 - \frac{m^2}{1-x} - \frac{m^2}{x}\right)\Big]^{-1} \\
&= \frac{e^2 g_s}{4\pi p^+ p^+}\int_{1-\alpha}^{1} dx \left(\frac{q'^2}{2(1-\alpha)} - \frac{M^2}{2} + \frac{m^2}{1-x}\right)\left(\frac{q'^2}{2(1-\alpha)} - M^2 + \frac{m^2}{1-x}\right) \\
&\cdot \left[\left(1-x-\alpha\right)\left(1-x\right)x\left(\frac{m^2}{1-x-\alpha} - \frac{m^2}{1-x} + M^2 - \frac{q'^2}{1-\alpha}\right)\left(\frac{m^2}{1-x} + \frac{m^2}{x} - M^2\right)\right]^{-1}
\end{aligned}
\tag{18}
$$

- However, this result gives

$$f_V^{(--)} + f_{NV}^{(--)} \neq F_{cov}$$

- Upon inspection, we realize that for this "minus-minus" component case, there is enough power of k^- on the numerator for the contour integration to have contribution from the arc at infinity
- When $k^- = Re^{i\theta} \to \infty$, the contribution is

$$\Gamma_D^{--} = \frac{ie^2 g_s}{4\pi^2} \lim_{R\to\infty} \int dk^+ \int_0^{\pm\pi} iRe^{i\theta} d\theta \frac{4(Re^{i\theta})^2}{2(p-k-q)^+ 2(p-k)^+ 2k^+ (Re^{i\theta})^3}$$

Subtracting the arc from the residue, we get

$$\begin{aligned}
&\Gamma^{--}_{D(b)} \\
&= \frac{e^2 g_s}{4\pi p^+ p^+} \int_0^{1-\alpha} dx \left\{ \left(\frac{M^2}{2} + \frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x} \right) \left(\frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x} \right) \right. \\
&\cdot \left[(1-x-\alpha)(1-x)x \left(\frac{m^2}{x} + \frac{m^2}{1-x-\alpha} - \frac{q'^2}{1-\alpha} \right) \left(\frac{m^2}{x} + \frac{m^2}{1-x} - M^2 \right) \right]^{-1} \\
&\left. - \frac{1}{2(1-x-\alpha)(1-x)x} \right\},
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\Gamma^{--}_{D(c)} \\
&= \frac{e^2 g_s}{4\pi p^+ p^+} \int_{1-\alpha}^{1} dx \left\{ \left(\frac{q'^2}{2(1-\alpha)} - \frac{M^2}{2} + \frac{m^2}{1-x} \right) \left(\frac{q'^2}{2(1-\alpha)} - M^2 + \frac{m^2}{1-x} \right) \right. \\
&\cdot \left[(1-x-\alpha)\,(1-x)\,x \left(\frac{m^2}{1-x-\alpha} - \frac{m^2}{1-x} + M^2 - \frac{q'^2}{1-\alpha} \right) \left(\frac{m^2}{1-x} + \frac{m^2}{x} - M^2 \right) \right]^{-1} \\
&\left. + \frac{1}{2(1-x-\alpha)(1-x)x} \right\}.
\end{aligned} \tag{20}$$

- But, this still results in disagreement between the “minus-minus” component calculation and the covariant one.
- In order to reveal the zero modes which may have been missed in the pure LFD calculation, next we try the same calculation in the interpolating dynamics between the LFD and IFD.
- To obtain results comparable to LFD results, we rotate the coordinates of the IFD close to but not exactly 45°, i.e., we take the interpolation method parameter $\mathbb{C} = \cos 2\delta$ small but non zero.

In the interpolation form, the 3 denominators of Eq. (15) can be rewritten as

$$\begin{aligned} D_1 = {} & \mathbb{C}\left(p_{\hat{+}} - k_{\hat{+}} - q_{\hat{+}}\right)^2 + 2\mathbb{S}\left(p_{\hat{+}} - k_{\hat{+}} - q_{\hat{+}}\right)\left(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}}\right) \\ & - \mathbb{C}\left(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}}\right)^2 - m^2 + i\varepsilon, \end{aligned} \tag{21}$$

$$D_2 = \mathbb{C}\left(p_{\hat{+}} - k_{\hat{+}}\right)^2 + 2\mathbb{S}\left(p_{\hat{+}} - k_{\hat{+}}\right)\left(p_{\hat{-}} - k_{\hat{-}}\right) - \mathbb{C}\left(p_{\hat{-}} - k_{\hat{-}}\right)^2 - m^2 + i\varepsilon, \tag{22}$$

and

$$D_3 = \mathbb{C}k_{\hat{+}}^2 + 2\mathbb{S}k_{\hat{+}}k_{\hat{-}} - \mathbb{C}k_{\hat{-}}^2 - m^2 + i\varepsilon. \tag{23}$$

Now there is no arc contribution because there is now 6 powers of the integration variable on the denominator, instead of 3, while still 3 on the numerator.

There are 6 poles in total

$$k_{\hat{+}1u,1d} = p_{\hat{+}} - q_{\hat{+}} + \frac{\mathbb{S}}{\mathbb{C}}(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}}) \mp \frac{\omega_1}{\mathbb{C}} \pm i\varepsilon \tag{24}$$

$$k_{\hat{+}2u,2d} = p_{\hat{+}} + \frac{\mathbb{S}}{\mathbb{C}}(p_{\hat{-}} - k_{\hat{-}}) \mp \frac{\omega_2}{\mathbb{C}} \pm i\varepsilon \tag{25}$$

$$k_{\hat{+}3d,3u} = -\frac{\mathbb{S}}{\mathbb{C}}k_{\hat{-}} \pm \frac{\omega_3}{\mathbb{C}} \mp i\varepsilon \tag{26}$$

where

$$\omega_1 = \sqrt{(p_{\hat{-}} - k_{\hat{-}} - q_{\hat{-}})^2 + \mathbb{C}m^2} \tag{27}$$

$$\omega_2 = \sqrt{(p_{\hat{-}} - k_{\hat{-}})^2 + \mathbb{C}m^2} \tag{28}$$

$$\omega_3 = \sqrt{k_{\hat{-}}^2 + \mathbb{C}m^2} \tag{29}$$

In the $\mathbb{C} \to 0$ limit, in each pair of the poles, one of them goes to infinity, the other goes to the light-front poles, Eq. (16).
Keeping the limit to the light-front in mind, we keep the relevant terms in the pole values at infinity taking limits such as $\mathbb{S} \to 1$, $k_{\underline{\hat{}}} \to k^+$ etc. already, while for the regular poles we substitute Eq. (16). The 6 poles become

$$k^-_{1reg} = p^- - q^- - \frac{m^2 - i\varepsilon}{2(p-k-q)^+} \tag{30}$$

$$k^-_{1inf} = \frac{p^+ - k^+ - q^+}{\mathbb{C}} \mp \frac{|p^+ - k^+ - q^+|}{\mathbb{C}} \pm i\varepsilon \tag{31}$$

$$k^-_{2reg} = p^- - \frac{m^2 - i\varepsilon}{2(p-k)^+} \tag{32}$$

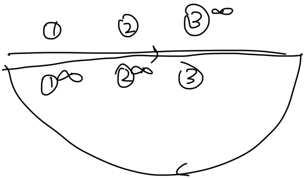
$$k^-_{2inf} = \frac{p^+ - k^+}{\mathbb{C}} \mp \frac{|p^+ - k^+|}{\mathbb{C}} \pm i\varepsilon \tag{33}$$

$$k^-_{3reg} = \frac{m^2 - i\varepsilon}{2k^+} \tag{34}$$

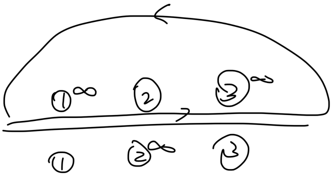
$$k^-_{3inf} = -\frac{k^+}{\mathbb{C}} \pm \frac{|k^+|}{\mathbb{C}} \mp i\varepsilon. \tag{35}$$

Now the pole structure becomes

Region (b) : $0 < x < 1-\alpha < 1$



Region (c) : $0 < 1-\alpha < x < 1$



Let us now calculate the residues of k^-_{1inf} and k^-_{2inf}. The residue of k^-_{1inf} is

$$\frac{ie^2 g_s}{4\pi^2}\int dk^+(-2\pi i)\frac{4}{2(p^+-k^+-q^+)(-2q^+)(2(p^+-q^+))} \tag{36}$$

The residue of k^-_{2inf} is

$$\frac{ie^2 g_s}{4\pi^2}\int dk^+(-2\pi i)\frac{4}{2(p^+-k^+)(2q^+)(2p^+)}. \tag{37}$$

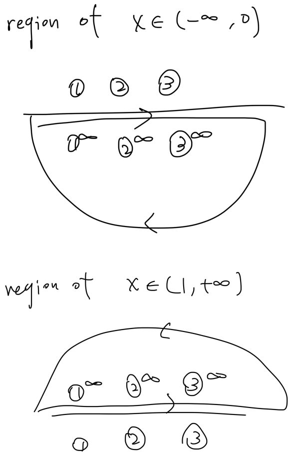
Adding the residues from all 3 poles on the lower half plane, we get

$$
\begin{aligned}
&\Gamma^{--}_{D(b)} \\
&= \frac{e^2 g_s}{4\pi p^+ p^+} \int_0^{1-\alpha} dx \left\{ \left(\frac{M^2}{2} + \frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x} \right) \left(\frac{q'^2}{2(1-\alpha)} - \frac{m^2}{x} \right) \right. \\
&\cdot \left[(1-x-\alpha)(1-x)x \left(\frac{m^2}{x} + \frac{m^2}{1-x-\alpha} - \frac{q'^2}{1-\alpha} \right) \left(\frac{m^2}{x} + \frac{m^2}{1-x} - M^2 \right) \right]^{-1} \\
&\left. + \frac{1}{(1-x-\alpha)(-\alpha)(1-\alpha)} + \frac{1}{(1-x)\alpha} \right\} . \qquad (38)
\end{aligned}
$$

Similarly, for region (c) we get

$$\Gamma^{--}_{D(c)} = \frac{e^2 g_s}{4\pi p^+ p^+}\int_{1-\alpha}^{1} dx \left\{ \left(\frac{q'^2}{2(1-\alpha)} - \frac{M^2}{2} + \frac{m^2}{1-x}\right)\left(\frac{q'^2}{2(1-\alpha)} - M^2 + \frac{m^2}{1-x}\right) \cdot \left[(1-x-\alpha)(1-x)x\left(\frac{m^2}{1-x-\alpha} - \frac{m^2}{1-x} + M^2 - \frac{q'^2}{1-\alpha}\right)\left(\frac{m^2}{1-x} + \frac{m^2}{x} - M^2\right)\right]^{-1} - \frac{1}{(1-x-\alpha)(-\alpha)(1-\alpha)} - \frac{1}{(-x)(1-\alpha)} \right\}. \quad (39)$$

- Now, adding these two time-ordered contributions still does not give agreement to the covariant result.
- The reason is, regions of x outside of $[0, 1]$ give contributions.
- The pole structures in those regions are



- These "poles at infinity" contributions must be included.

Calculating their residues, we obtain for the region of $x < 0$,

$$\Gamma^{--}_{D(x\in(-\infty,0))} = \frac{e^2 g_s}{4\pi p^+ p^+} \int_{-\infty}^{0} dx \left\{ \frac{1}{(1-x-\alpha)(-\alpha)(1-\alpha)} + \frac{1}{(1-x)\alpha} + \frac{1}{(-x)(1-\alpha)} \right\}. \tag{40}$$

Similarly for the region of $x > 1$, we obtain

$$\Gamma^{--}_{D(x\in(1,+\infty))} = \frac{e^2 g_s}{4\pi p^+ p^+} \int_{1}^{+\infty} dx \left\{ -\frac{1}{(1-x-\alpha)(-\alpha)(1-\alpha)} - \frac{1}{(1-x)\alpha} - \frac{1}{(-x)(1-\alpha)} \right\}. \tag{41}$$

Now, adding all 4 regions, we finally reached agreement between the "minus-minus" component form factor calculation and the covariant one.

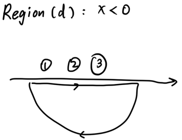
As it turns out, in the pure LFD calculation, the reason for the previous disagreement was also because of missing the $x < 0$ and $x > 1$ regions. These regions usually do not contribute because all the poles are on the same side, and one can take the half plane which has no poles in it, however, for the "$--$" component, there is arc contribution, so the regions of $x < 0$ and $x > 1$ do contribute.

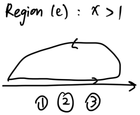
The arc gives

$$\Gamma^{--}_{D(d)} = -\frac{e^2 g_s}{4\pi p^+ p^+}\int_{-\infty}^{0} dx \frac{1}{2(1-x-\alpha)(1-x)x},$$

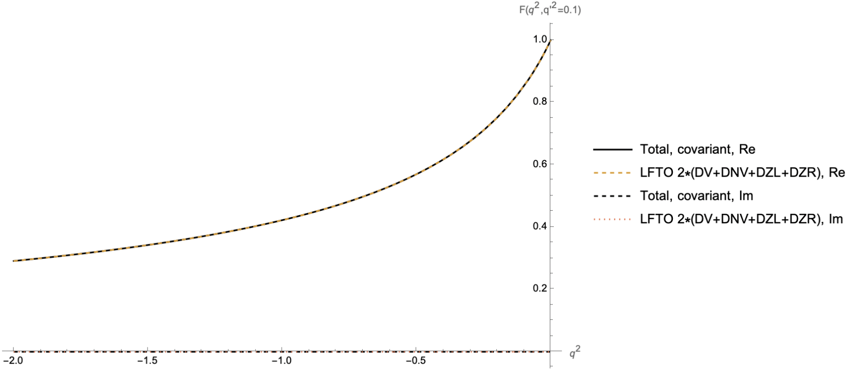
and

$$\Gamma^{--}_{D(e)} = \frac{e^2 g_s}{4\pi p^+ p^+}\int_{1}^{+\infty} dx \frac{1}{2(1-x-\alpha)(1-x)x}.$$





Adding (b), (c), (d), and (e) all together, again agreement with the covariant result is achieved.


F(q^2,q'^2=0.1)
1.0
0.8
0.6
0.4
0.2
−2.0
−1.5
−1.0
−0.5
q^2
Total, covariant, Re
LFTO 2*(DV+DNV+DZL+DZR), Re
Total, covariant, Im
LFTO 2*(DV+DNV+DZL+DZR), Im

Old-fashioned time-ordered perturbation theory

with interpolation method

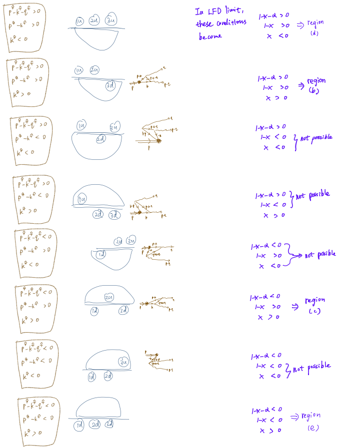
In the old-fashioned time-ordered perturbation theory, in the interpolation form, one can write the 3 denominators as

$$\frac{1}{D_1 D_2 D_3} = \frac{1}{-2\omega_1}\left(\frac{1}{k^{\hat{-}}_{+} - k^{\hat{-}}_{+1u}} - \frac{1}{k^{\hat{-}}_{+} - k^{\hat{-}}_{+1d}}\right) \times \frac{1}{-2\omega_2}\left(\frac{1}{k^{\hat{-}}_{+} - k^{\hat{-}}_{+2u}} - \frac{1}{k^{\hat{-}}_{+} - k^{\hat{-}}_{+2d}}\right) \times \frac{1}{-2\omega_3}\left(\frac{1}{k^{\hat{-}}_{+} - k^{\hat{-}}_{+3u}} - \frac{1}{k^{\hat{-}}_{+} - k^{\hat{-}}_{+3d}}\right), \quad (42)$$

where the energy poles are given in Eqs. (24), (25), and (26), and the ω's are given in Eqs. (27), (28), and (29).

By expanding the parenthesis, one gets 8 time-ordered diagrams, among which 2 normally do not contribute, because of all the poles on same side, however in this case they do contribute because of the arc.

$$
\begin{aligned}
\frac{1}{D_1 D_2 D_3} = &\frac{1}{-2\omega_1}\frac{1}{-2\omega_2}\frac{1}{-2\omega_3} \\
\times \Bigg(&\frac{1}{k_{\hat{+}} - k_{\hat{+}1u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3u}} \\
- &\frac{1}{k_{\hat{+}} - k_{\hat{+}1u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3d}} \\
- &\frac{1}{k_{\hat{+}} - k_{\hat{+}1u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3u}} \\
+ &\frac{1}{k_{\hat{+}} - k_{\hat{+}1u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3d}} \\
- &\frac{1}{k_{\hat{+}} - k_{\hat{+}1d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3u}} \\
+ &\frac{1}{k_{\hat{+}} - k_{\hat{+}1d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2u}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3d}} \\
+ &\frac{1}{k_{\hat{+}} - k_{\hat{+}1d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3u}} \\
- &\frac{1}{k_{\hat{+}} - k_{\hat{+}1d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}2d}}\frac{1}{k_{\hat{+}} - k_{\hat{+}3d}} \Bigg).
\end{aligned}
$$



As an example, the first diagram can be calculated as the following which is solely arc contribution

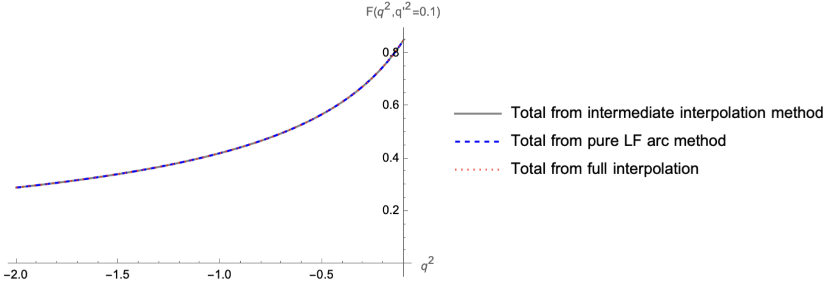
$$\Gamma^{\hat{-}\hat{-}}_{D(d)} = -\frac{ie^2 g_s}{4\pi^2} \lim_{R\to\infty} \int dk_{\hat{-}} \int_0^{-\pi} iRe^{i\theta} d\theta \frac{4(Re^{i\theta})^2}{-8\omega_1\omega_2\omega_3 (Re^{i\theta})^3}. \quad (43)$$

And the second diagram can be calculated as the following doing the pole integration and also accounting for the arc contribution

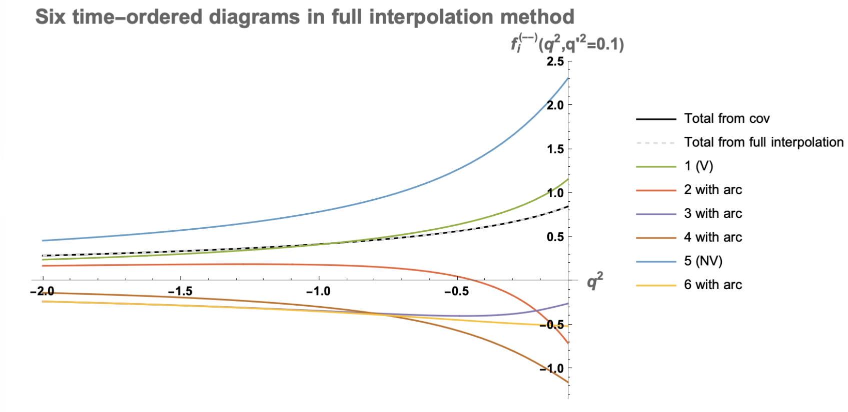
$$\begin{aligned} \Gamma^{\hat{-}\hat{-}}_{D(b)} = & \frac{ie^2 g_s}{4\pi^2} \int dk_{\hat{-}} \frac{1}{(k_{\hat{+}3d} - k_{\hat{+}1u})(k_{\hat{+}3d} - k_{\hat{+}2u})} \times \Big[4(k_{\hat{+}3d})^2 \\ & -2\left(p'^{\hat{+}} - p_{\hat{-}} + 2q'^{\hat{+}} - 2q'_{\hat{-}}\right) k_{\hat{+}3d} + \left(p^{\hat{+}} - p_{\hat{-}} + q'^{\hat{+}} - q'_{\hat{-}}\right)\left(q'^{\hat{+}} - q'_{\hat{-}}\right)\Big] \\ & - \frac{ie^2 g_s}{4\pi^2} \lim_{R\to\infty} \int dk_{\hat{-}} \int_0^{-\pi} iRe^{i\theta} d\theta \frac{4(Re^{i\theta})^2}{-8\omega_1\omega_2\omega_3 (Re^{i\theta})^3}. \end{aligned} \quad (44)$$

Similarly, all the 8 diagrams are calculated.

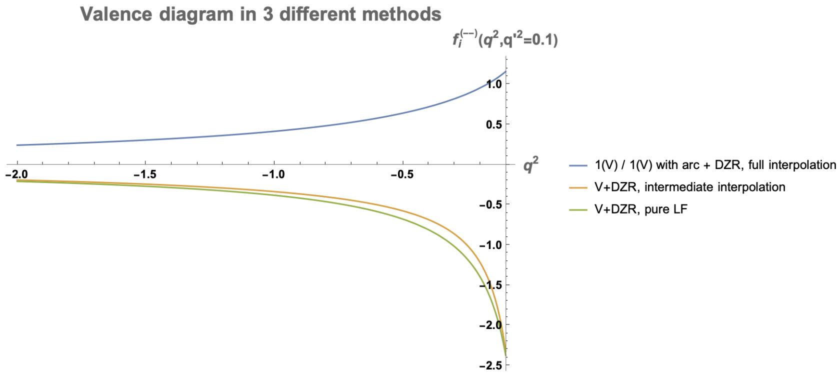
- Diagrams (d) and (e) cancelled with the arc contributions in diagrams (b) and (c).
- The other 4 diagrams each has arc contributions which add up in the total answer.
- The 4 time-ordered diagrams which vanish kinematically in the light-front dynamics persist when taking the limit to the light-front.
- The total answer agrees with the intermediate interpolation method calculation, the pure LFD calculation, as well as the covariant result.


F(q^2,q'^2=0.1)
0.8
0.6
0.4
0.2
-2.0
-1.5
-1.0
-0.5
q^2
Total from intermediate interpolation method
Total from pure LF arc method
Total from full interpolation

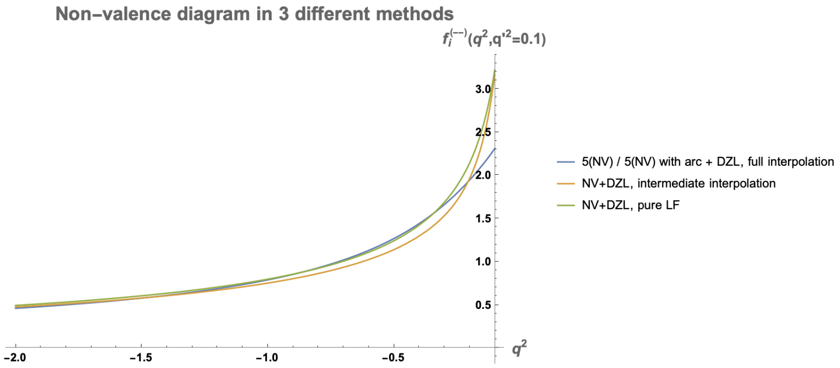
Results from the full interpolation method



Comparing the results from the 3 different methods of taking the limit to LFD



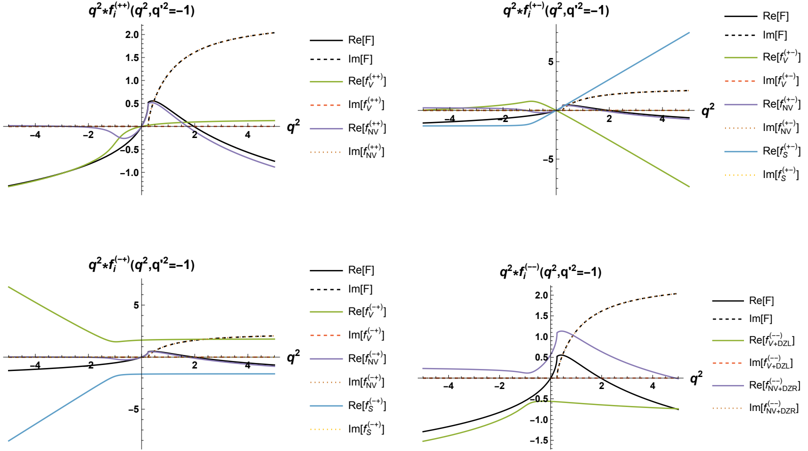
Comparing the results from the 3 different methods of taking the limit to LFD



Spurious form factors in the light-front dynamics

Individual LFTO form factors depend on the component

Taking $q'^2 = -1.0 GeV^2$.



Spurious form factors

The most general way, is to write the LFTO amplitudes as 4 form factors

$$\Gamma_i^{\mu\nu} = f_i^A(q^2, q'^2)A^{\mu\nu} + f_i^B(q^2, q'^2)B^{\mu\nu} + f_i^C(q^2, q'^2)C^{\mu\nu} + f_i^D(q^2, q'^2)D^{\mu\nu}.$$

The four forms are found to be

$$\begin{aligned} A^{\mu\nu} &= g^{\mu\nu} q \cdot q' - q'^{\mu} q^{\nu}, \\ B^{\mu\nu} &= q^{\mu} q'^{\nu}, \\ C^{\mu\nu} &= q^{\mu}\left(q^{\nu} - \frac{q \cdot q'}{q'^2} q'^{\nu}\right), \\ D^{\mu\nu} &= \left(q'^{\mu} - \frac{q \cdot q'}{q^2} q^{\mu}\right) q'^{\nu}. \end{aligned}$$

where we select them so that each two out of the four are orthogonal.
Only $A^{\mu\nu}$ is gauge invariant, while $B^{\mu\nu}$, $C^{\mu\nu}$, and $D^{\mu\nu}$ are not.
Thus, they must satisfy

$$\sum_i f_i^B(q^2, q'^2) = \sum_i f_i^C(q^2, q'^2) = \sum_i f_i^D(q^2, q'^2) = 0. \qquad (45)$$

(Now the component-dependence is completely in those forms).

We can obtain the individual form factors by

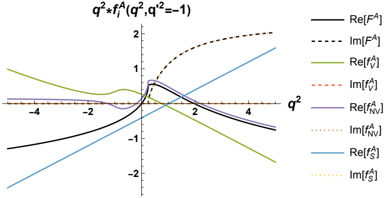
$$f_i^A(q^2, q'^2) = \frac{A_{\mu\nu}\Gamma_i^{\mu\nu}}{A_{\mu\nu}A^{\mu\nu}} = \frac{A_{\mu\nu}\Gamma_i^{\mu\nu}}{q^2 q'^2} \tag{46}$$

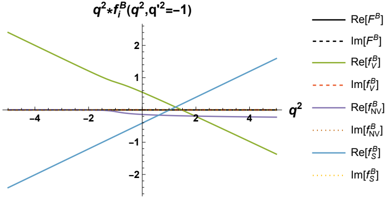
$$f_i^B(q^2, q'^2) = \frac{B_{\mu\nu}\Gamma_i^{\mu\nu}}{B_{\mu\nu}B^{\mu\nu}} = \frac{B_{\mu\nu}\Gamma_i^{\mu\nu}}{q^2 q'^2} \tag{47}$$

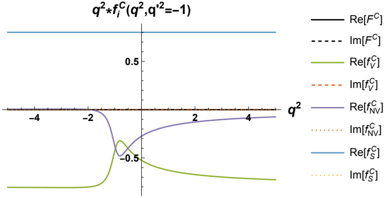
$$f_i^C(q^2, q'^2) = \frac{C_{\mu\nu}\Gamma_i^{\mu\nu}}{C_{\mu\nu}C^{\mu\nu}} = \frac{C_{\mu\nu}\Gamma_i^{\mu\nu}}{q^2 \left(q^2 - \frac{(q\cdot q')^2}{q'^2}\right)} \tag{48}$$

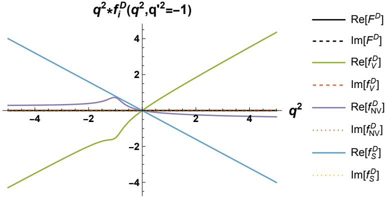
$$f_i^D(q^2, q'^2) = \frac{D_{\mu\nu}\Gamma_i^{\mu\nu}}{D_{\mu\nu}D^{\mu\nu}} = \frac{D_{\mu\nu}\Gamma_i^{\mu\nu}}{q'^2 \left(q'^2 - \frac{(q\cdot q')^2}{q^2}\right)}. \tag{49}$$

Results for all 4 form factors









Summary

- One can compute not only the good current, but also the bad current in the LFD, in order to define the individual LF time-ordered contributions to the form factor.
- In the computation of the “minus-minus” component of the current, special care is needed to take into account the contributions from regions of x outside of the usual $[0, 1]$.
- The individual LF time-ordered amplitudes depend on the component, thus are not gauge invariant, and 3 spurious form factors arise in this $1 + 1$-d scalar model case.

Thank you for your attention!